Differential Calculus in Braided Abelian Categories

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Abstract

Braided non-commutative differential geometry is studied. In particular we investigate the theory of (bicovariant) differential calculi in braided abelian categories. Previous results on crossed modules and Hopf bimodules in braided categories are used to construct higher order bicovariant differential calculi over braided Hopf algebras out of first order ones. These graded objects are shown to be braided differential Hopf algebras with universal bialgebra properties. The article especially extends Woronowicz's results on (bicovariant) differential calculi to the braided non-commutative case.

Keywords: Braided category, Hopf algebra, Hopf bimodule, Differential calculus

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Introduction

Differential geometry and group theory interact very fruitfully within the theory of Lie groups. Tangent Lie algebras, invariant differential forms, infinitesimal representations, principal bundles, gauge theory, etc. emerged out of this interplay. It was discovered by Woronowicz that many of these geometrically related structures can be generalized to non-commutative geometry [Wor]. He introduced and studied the differential calculus on compact quantum groups or more generally on Hopf algebras over a field $\Bbbk$ with $\mathrm{char}\,(\Bbbk) = 0$. His theory is built on the base category of $\Bbbk$-modules with the usual tensor product and the involutive tensor transposition $\tau : a \otimes b \mapsto b \otimes a$. In what follows we refer to the conditions in [Wor] as the classical conditions in contrast to our investigations in the braided case. There have been a lot of publications along the classical lines of [Wor]. However, it would be beyond the scope of this introduction to give an appropriate appreciation to all of them. Nevertheless we would like to mention three articles [Brz, Mal, Man] besides [Wor] which particularly influenced our work from the differential geometrical point of view in a considerable way. The differential graded algebra approach to quantum groups and the quantum de Rham complexes were studied in [Mal, Man]. In Manin's work [Man, Proposition 2.6.1] very general conditions are found under which an operator ring (of differentials) over an algebra has a bialgebra structure. The non-commutative differential calculus discovered by Woronowicz is indeed of this type. The differential Hopf algebra structure of the higher order differential calculi of [Wor] has been unfolded explicitely in [Brz]. Essentially all of these articles proceed on the assumption that the classical symmetric conditions are at the bottom of the theory of (bicovariant) differential calculi.

Braided or quasisymmetric monoidal categories had been initially investigated by Joyal and Street in [JS]. Footing on their work, Majid generalized the notion of (tensor) algebras, bi- and Hopf algebras to categories with a

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tensor product which has a not necessarily involutive commutation behaviour expressed by the so-called braiding. The braiding takes over the rôle of the tensor transposition τ. As standard references to Majid's investigations on this subject we refer to [Ma1, Ma6] where many foundations of braided mathematics and in particular of braided geometry can be found.

In our article we are interested in the amalgamation of differential geometrical methods and braided category theory leading to *braided non-commutative differential geometry*. It extends the results of [BD2]. Several successful steps in bringing together both methods have been done for instance in [BM, IV, Ma4]. We show that the theory of differential calculus of Lie groups and of Hopf algebras can be generalized to braided non-commutative differential geometry. We examine (bicovariant) differential calculi over Hopf algebras in a braided abelian category. Nowhere in the main results of our article we suppose the objects under consideration to be sets. Our investigation is footed completely on categorical language which allows us to lay bare fundamental structures of the theory of differential calculus. Working with these key-stones we are able to build in the theory into the framework of a very general class of braided abelian categories. Our categorical set-up is general enough to cover and extend all known examples of categories where braided differential geometry has already been applied. On the other side it is rich enough to obtain a theory of braided (bicovariant) differential calculus including the notations of [Mal, Man, Wor]. We show that every first order bicovariant differential calculus over a braided Hopf algebra induces in a natural way a higher order bicovariant differential calculus. It is a differential Hopf algebra with certain universal properties – this generalizes results of [Brz, Mal, Man] to braided categories. Especially, our results can be applied to the study of (bicovariant) differential calculi of quantum planes [FRT, Ma4, Ma5] and of braided groups [Ma6] but also for the investigation of differential geometric structures of, say, degenerate Sklyanin algebras [Ma9] and other interesting objects in braided categories. In connection with [BD1, Bes] bicovariant differential calculi of cross product Hopf algebras can be investigated in relation to the differential calculi of their particular factors. Our results may also contribute to the investigation of braided gauge theory and quantum bundle theory [BM].

The article is devided into 4 chapters. Chapter 1 mainly reviews and reformulates definitions and results of [BD1] which are necessary for the understanding of the subsequent chapters. The proofs of the particular statements can be found in [BD1]. The isomorphism theorem of Hopf bimodule bialgebras and bialgebra projections in braided monoidal categories is of special importance. It initiated our investigations on braided bicovariant differential calculi. Hopf bimodules or bicovariant bimodules are the fundamental objects for the definition of bicovariant differential calculi over a Hopf algebra. The isomorphism theorem is an essential tool for the construction of higher order differential calculi.

In Chapter 2 we fix our categorical basis and state essential results on it. We describe modules generated by morphisms which will be useful in particular for the definition and investigation of differential calculi in Chapter 4. Furthermore we are concerned with factor (co-)algebras and bialgebras arising from canonical epi-mono decompositions of (co-)algebra and bialgebra morphisms respectively. Later in Chapter 3 this decomposition allows us to derive very naturally the antisymmetric and exterior Hopf algebra objects needed for the construction of higher order differential calculi. A further restriction of the properties of the categories under consideration allows us to use a definition of (co-)ideals and bi-ideals in such categories and to establish well known results of ring theory in our more general context. This will be applied in Chapters 3 and 4.

In Chapter 3 we investigate graded categories. We are using the notion of a coalgebra category [CF] to study under which conditions graded categories, or more general, functor categories are monoidal. We introduce a quasitriangular structure on a coalgebra category and show that the corresponding functor categories are braided in this case. As a special case we are interested in graded categories and categories of complexes over $\{0, 1\}$ and $\mathbb{N}_0$. We investigate (co-)algebras, bi- and Hopf algebras in such categories. We introduce braided combinatorics in

the gereralized sense of [Ma4, Wor]. Using the results of Chapters 1 and 2 we construct analogues of antisymmetric tensor Hopf algebras over an arbitrary object in the braided category $\mathcal{C}$. For a Hopf bimodule X over a Hopf algebra H in $\mathcal{C}$ we build the exterior Hopf algebra of forms $X^{\wedge_H}$. This object is the braided version of Woronowicz's so-called external algebra $\Gamma^\wedge$ over a bicovariant bimodule Γ [Wor]. It is universal as a bialgebra generated algebraically by its components of order 0 and 1.

Finally Chapter 4 is concerned with differential calculi. In the sense of [Mal, Man, Wor] we define a differential calculus as a differential graded algebra which is generated by its 0th component and the image of it under the differential d_0. This generalized point of view is due to [Mal, Man]. From [Man, Chapters 0 and 2] and [Mal, Theorem 1.2.3] one can see that both first and higher order differential calculi of [Wor] fit into this scheme. Manin calls the differential calculi quantum de Rham complexes. For any graded bialgebra $\hat{B}$ in the braided category we define $\hat{B}$-left, $\hat{B}$-right and $\hat{B}$-bicovariant differential calculi as the corresponding $\hat{B}$-comodule differential calculi. In particular we examine braided bicovariant (first order) differential calculi (X, d) over a Hopf algebra H. We define the (unique) maximal differential calculus of a given differential algebra and study its structure. Using comma category technique we establish a braided Woronowicz construction of a higher order differential Hopf algebra out of a given first order bicovariant differential calculus. From this object we extract the exterior Hopf algebra of differential forms $(X^{\wedge_H}, \mathrm{d}^\wedge)$ over H as its maximal differential calculus. $(X^{\wedge_H}, \mathrm{d}^\wedge)$ coincides with the first order calculus (X, d) in the 0th and 1st component and is therefore its unique higher order extension as a differential calculus.

The appendix provides facts from classical category theory connected with the present article and [BD1]. Mostly we will work with these results without reference.

1 Preliminaries and Notations

We review results and techniques from [BD1, JS, Ma2, Ma6] essential for our purposes. We assume that the reader is familiar with the notion of a braided monoidal category $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \rho, \lambda, \Psi)$ [FY, JS, Ma1]. If not otherwise stated we suppose henceforth that $\mathcal{C}$ is braided and admits split idempotents. I.e. for every idempotent $e = e^2 : X \longrightarrow X$ in $\mathcal{C}$ there exist morphisms $\mathrm{i}_e : X_e \longrightarrow X$, $\mathrm{p}_e : X \longrightarrow X_e$ in $\mathcal{C}$, such that $e = \mathrm{i}_e \circ \mathrm{p}_e$ and $\mathrm{id}_e = \mathrm{p}_e \circ \mathrm{i}_e$. This is not a severe restriction, because every (braided monoidal) category can be canonically embedded into a (braided monoidal) category that admits split idempotents. Abelian categories automatically admit split idempotents.

The notations of (co-)algebras, bi- and Hopf algebras, (co-)modules and bi-(co-)modules can be generalized obviously to the category $\mathcal{C}$ [Ma1, Ma2, Ma6]. Loosely speaking we replace in the defining relations of the objects the tensor transposition τ by the braiding Ψ. Since we are dealing with braided categories we occasionally use graphical presentation of morphisms. The graphical calculus turns out to be a very convenient tool for dealing with complicated expressions of morphisms. There are several detailed expositions on this subject [FY, Lyu, Ma1, Ma2, Yet]. We are using the notation of [BD1]. In the article we will sometimes obtain statements simply by dualizing previous results. This means that we consider the original outcome in the opposite category and transpose it to its dual version in the category. In diagrammatic language this may be interpreted as reversing the arrows. The main results of [BD1] which we need in the sequel are concerned with Hopf bimodules and crossed modules in braided categories. Hopf bimodules or bicovariant bimodules in symmetric categories are the essential objects in the definition of bicovariant differential calculi. Crossed modules are related to the invariant vector fields of a bicovariant differential calculus [Wor]. They appear naturally in the representation theory of quantum groups [Yet]. In the braided case Hopf bimodules and crossed modules will turn out to be of similar importance for the

Hopf bimodule identity Crossed module identity

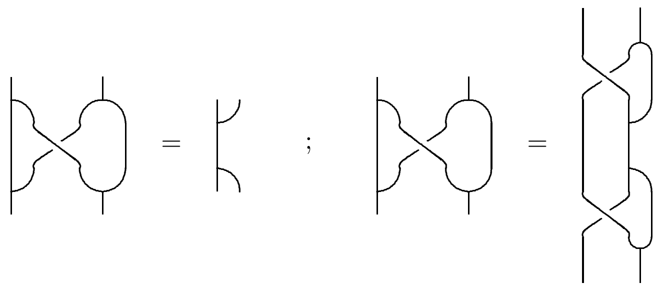

Figure 1: Some defining relations for Hopf bimodules and crossed modules

investigation of braided bicovariant differential calculi. We recall the definition of Hopf bimodules and crossed modules in braided categories [BD1]

DEFINITION 1.1 *Let $(B, \mathrm{m}, \eta, \Delta, \varepsilon)$ be a bialgebra in $\mathcal{C}$. An object $(X, \mu_r, \mu_l, \nu_r, \nu_l)$ is called a B-Hopf bimodule if (X, μ_r, μ_l) is a B-bimodule, and if (X, ν_r, ν_l) is a B-bicomodule in the category of B-bimodules where the regular action m on B and the diagonal action on tensor products of modules are used. B-Hopf bimodules together with the B-bimodule-B-bicomodule morphisms form a category which will be denoted by ${}^B_B\mathcal{C}^B_B$.*

The diagonal action of the tensor product of, say, two B-left modules X and Y is given by

$$\mu_l^{X\otimes Y} = (\mu_l^X \otimes \mu_l^Y) \circ (\mathrm{id}_B \otimes \Psi_{B,X} \otimes \mathrm{id}_Y) \circ (\Delta \otimes \mathrm{id}_X \otimes \mathrm{id}_Y)\,. \tag{1.1}$$

Similarly the diagonal action of right modules is defined.

DEFINITION 1.2 *A right crossed module (X, μ_r, ν_r) over the Hopf algebra H in $\mathcal{C}$ is an H-right module and an H-right comodule obeying the compatibility relations*

$$\begin{aligned}(\mathrm{id}_X \otimes m) \circ (\Psi_{H,X} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \nu_r \circ \mu_r) \circ (\Psi_{X,H} \otimes H) \circ (\mathrm{id}_X \otimes \Delta)\\ = (\mu_r \otimes m) \circ (\mathrm{id}_X \otimes \Psi_{H,H} \otimes \mathrm{id}_H) \circ (\nu_r \otimes \Delta).\end{aligned} \tag{1.2}$$

The category $\mathcal{DY}(\mathcal{C})^H_H$ is the category of crossed modules. The morphisms are right-module-right-comodule morphisms over H. In a similar way all other combinations of crossed modules will be defined.

In the first part of Figure 1 we exemplarily represent the right module morphism property of the right coaction of a Hopf bimodule. The defining identity (1.2) of a crossed module is represented graphically in the second part of Figure 1.

EXAMPLE 1 *A Hopf algebra H is a crossed module H_{ad} over itself through the right adjoint action*

$$\mu_r^{\mathrm{ad}} := \mu_l \circ (\mathrm{id}_H \otimes \mu_r) \circ (\Psi_{X,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_X \otimes (S \otimes \mathrm{id}_H) \circ \Delta) \tag{1.3}$$

and the comultiplication Δ as regular coaction. Dually H is a crossed module H^{ad} through the regular action and the right coadjoint coaction.

The following description of the functor ${}_H(\_)$ holds for Hopf modules in general. However we restrict to Hopf bimodules here for conceptional reasons. We state the corresponding results of [BD1] in invariant form. If H is a Hopf algebra in the category $\mathcal{C}$ and $(X, \mu_r, \mu_l, \nu_r, \nu_l)$ is an H-Hopf bimodule then there exists an object ${}_H X$ such that ${}_H X \cong \mathbf{1} \underset{H}{\otimes} X$, ${}_H X \cong \mathbf{1} \underset{H}{\Box} X$ and ${}_X\mathrm{p} \circ {}_X\mathrm{i} = \mathrm{id}_{{}_H X}$ where ${}_X\mathrm{p} : \mathbf{1} \otimes X \cong X \longrightarrow {}_H X \cong \mathbf{1} \underset{H}{\otimes} X$ and ${}_X\mathrm{i} : \mathbf{1} \underset{H}{\Box} X \cong {}_H X \longrightarrow X \cong \mathbf{1} \otimes X$ are the corresponding universal morphisms. This means in particular that ${}_H X$ is at the same time tensor product and cotensor product over H of $\mathbf{1}$ and X. It is unique up to isomorphism, ${}_X\mathrm{p}' = \xi \circ {}_X\mathrm{p}$ and ${}_X\mathrm{i}' = {}_X\mathrm{i} \circ \xi^{-1}$. The object of (co-)invariants ${}_H X$ is a crossed module. The assignment ${}_H(\_) : {}^H_H\mathcal{C}^H_H \longrightarrow \mathcal{DY}(\mathcal{C})^H_H$, which is given through ${}_H(X) := ({}_H X, {}_X\mathrm{p} \circ \mu_r \circ ({}_X\mathrm{i} \otimes \mathrm{id}_H), ({}_X\mathrm{p} \otimes \mathrm{id}_H) \circ \nu_r \circ {}_X\mathrm{i})$ for an object X, and through ${}_H(f) = {}_Y\mathrm{p} \circ f \circ {}_X\mathrm{i}$ for a Hopf bimodule morphism $f : X \longrightarrow Y$, defines a functor. It is a braided monoidal equivalence as will be described below in Theorem 1.4.

For the category of Hopf bimodules over a Hopf algebra H a tensor bifunctor can be defined such that ${}^H_H\mathcal{C}^H_H$ is a (braided) monoidal category [BD1]. We will give an invariant formulation of this fact in the following theorem. As pre-requisites we define two bifunctors $\odot$ and $\boxdot$ on the category ${}^H_H\mathcal{C}^H_H$ over a Hopf algebra H. Two objects X and Y of ${}^H_H\mathcal{C}^H_H$ yield the H-Hopf bimodule $X \odot Y$ with diagonal left and right actions $\mu^{X\odot Y}_{d,l}$ and $\mu^{X\odot Y}_{d,r}$ according to (1.1), and with induced left and right coactions $\nu^{X\odot Y}_{i,l} = \nu^X_l \otimes \mathrm{id}_Y$ and $\nu^{X\odot Y}_{i,r} = \mathrm{id}_X \otimes \nu^Y_r$. The Hopf bimodule $X \boxdot Y$ is obtained in the dual symmetric manner. For Hopf bimodule morphisms f and g we define $f \odot g = f \boxdot g = f \otimes g$. Then the categories $({}^H_H\mathcal{C}^H_H, \odot, \alpha)$ and $({}^H_H\mathcal{C}^H_H, \boxdot, \alpha)$ are semi-monoidal, i. e. they are categories which are almost monoidal, except that the unit object and the relations involving it are not required. One verifies easily that the identity functor $\mathrm{id}_{{}^H_H\mathcal{C}^H_H} : \left({}^H_H\mathcal{C}^H_H, \boxdot\right) \longrightarrow \left({}^H_H\mathcal{C}^H_H, \odot^{\mathrm{op}}\right)$ is semi-monoidal with respect to the natural transformation $\Theta : \boxdot \xrightarrow{\bullet} \odot^{\mathrm{op}}$ defined through

$$\Theta_{X,Y} := (\mu^Y_l \otimes \mu^X_r) \circ (\mathrm{id}_H \otimes \Psi_{X,Y} \otimes \mathrm{id}_H) \circ (\nu^X_l \otimes \nu^Y_r) \; : \; X \otimes Y \longrightarrow Y \otimes X \, . \tag{1.4}$$

Theorem 1.3 *Let H be a Hopf algebra in $\mathcal{C}$. The category ${}^H_H\mathcal{C}^H_H$ of Hopf bimodules over H is monoidal with the regular Hopf bimodule H as unit object and with the tensor product $\otimes_H$ uniquely defined (up to monoidal equivalence) by one of the following equivalent conditions for any pair of H-Hopf bimodules X and Y.*

- *The H-Hopf bimodule $X \otimes_H Y$ is the tensor product over H of the underlying modules, and the canonical morphism $\lambda^H_{X,Y} : X \odot Y \longrightarrow X \otimes_H Y \cong X \underset{H}{\otimes} Y$ is functorial in ${}^H_H\mathcal{C}^H_H$, i. e. $\lambda^H : \odot \xrightarrow{\bullet} \underset{H}{\otimes}$.*
- *The H-Hopf bimodule $X \otimes_H Y$ is the cotensor product over H of the underlying comodules, and the canonical morphism $\rho^H_{X,Y} : X \underset{H}{\Box} Y \cong X \otimes_H Y \longrightarrow X \boxdot Y$ is functorial in ${}^H_H\mathcal{C}^H_H$, i. e. $\rho^H : \underset{H}{\Box} \xrightarrow{\bullet} \boxdot$.*

Then for given natural morphisms λ^H and ρ^H it holds

$$\rho^H_{X,Y} \circ \lambda^H_{X,Y} = (\mu^X_r \otimes \mu^Y_l) \circ (\mathrm{id}_X \otimes \Psi_{H\,H} \otimes \mathrm{id}_Y) \circ (\nu^X_r \otimes \nu^Y_l) \, . \tag{1.5}$$

The category ${}^H_H\mathcal{C}^H_H$ is pre-braided through the pre-braiding ${}^{{}^H_H\mathcal{C}^H_H}\Psi_{X,Y}$ uniquely defined by the condition

$$\rho^H_{Y,X} \circ {}^{{}^H_H\mathcal{C}^H_H}\Psi_{X,Y} \circ \lambda^H_{X,Y} = \Theta_{X,Y} \, . \tag{1.6}$$

${}^H_H\mathcal{C}^H_H$ is braided if the antipode of H is an isomorphism.

Proof. The existence of a braided monoidal structure on ${}^H_H\mathcal{C}^H_H$ obeying the conditions of Theorem 1.3 has been deduced explicitely in [BD1]with the help of the functor ${}_H(\_)$. The universal morphisms for the tensor product and the cotensor product are

$$\begin{aligned}&\lambda^H_{0\,N,M} : N \otimes M \longrightarrow N \underset{H}{\otimes} M \cong N \otimes {}_H(M)\\&\lambda^H_{0\,N,M} = (\mu^N_r \otimes \mathrm{id}_{HM}) \circ (\mathrm{id}_N \otimes (\mathrm{id}_H \otimes {}_M\mathrm{p}) \circ \nu^M_l)\end{aligned}\tag{1.7}$$

and

$$\begin{aligned}&\rho^H_{0\,N,M} : N \otimes {}_H(M) \cong N \underset{H}{\square} M \longrightarrow N \otimes M\\&\rho^H_{0\,N,M} = (\mathrm{id}_N \otimes \mu^M_l \circ (\mathrm{id}_H \otimes {}_M\mathrm{i})) \circ (\nu^N_r \otimes \mathrm{id}_{HM})\,.\end{aligned}\tag{1.8}$$

Assume now that there exists another natural transformation $\lambda^H : \odot \xrightarrow{\bullet} \underset{H}{\otimes}$ obeying the first condition of the theorem. Once such a morphism λ^H is given, the induced Hopf bimodule structure on the objects $X \otimes_H Y$ is uniquely determined, since $\lambda^H_{X,Y}$ and therefore $\lambda^H_{H\square X,Y} \cong \mathrm{id}_H \otimes \lambda^H_{X,Y}$ are Hopf bimodule epimorphisms. Then there are functorial H-Hopf bimodule isomorphisms ξ between different tensor products, $\xi_{X,Y} : X \otimes {}_H(Y) \longrightarrow X \otimes_H Y$, such that $\xi_{X,Y} \circ \lambda^H_{0\,X,Y} = \lambda^H_{X,Y}$. Then the second condition of the theorem is immediately verified by defining $\rho^H_{X,Y} := \rho^H_{0\,X,Y} \circ \xi_{X,Y}$. In a canonical way the pre-braided monoidal structure induced by λ^H can be established. Starting with a natural morphism $\rho^H : \underset{H}{\square} \xrightarrow{\bullet} \boxdot$ obeying the second condition of the theorem, analogous results are obtained in the dual manner. ■

Remark 1 If the antipode S of H is an isomorphism, the inverse ${}^{{}^H_H\mathcal{C}^H_H}\Psi^{-1}$ of the braiding ${}^{{}^H_H\mathcal{C}^H_H}\Psi$ is given by

$${}^{{}^H_H\mathcal{C}^H_H}\Psi^{-1}{}_{X,Y} = \lambda^H_{Y,X} \circ (\mu^Y_r \circ \Psi^{-1}{}_{H,Y} \otimes \mathrm{id}_X) \circ (S^{-1} \otimes \Psi^{-1}{}_{X,Y}) \circ (\Psi^{-1}{}_{X,H} \circ \nu^X_r \otimes \mathrm{id}_Y) \circ \rho^H_{X,Y}\,. \tag{1.9}$$

In a symmetric way (1.9) can be expressed analogously through the left (co-)actions μ^X_l and ν^Y_l because of the universal property of the (co-)tensor product. The proof of (1.9) is based on the following observation. For $f : X \underset{H}{\otimes} Y \longrightarrow U \underset{H}{\square} V$ we put $\tilde{f} := \rho^H_{U,V} \circ f \circ \lambda^H_{X,Y}$. Then $\widetilde{fg} = \tilde{f} \circ (\mu_r \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\nu_r \otimes \mathrm{id}) \circ \tilde{g}$.

We outline the correspondence of Hopf bimodules and crossed modules in braided categories (with split idempotents). A full inclusion functor of the category of H-right crossed modules into the category of H-Hopf bimodules, $H \ltimes (\_) : \mathcal{DY}(\mathcal{C})^H_H \longrightarrow {}^H_H\mathcal{C}^H_H$, is defined by $H \ltimes (X) = (H \otimes X, \mu^{H\otimes X}_{i,l}, \nu^{H\otimes X}_{i,l}, \mu^{H\otimes X}_{d,r}, \nu^{H\otimes X}_{d,r})$ for any right crossed module X and by $H \ltimes (f) = \mathrm{id}_H \otimes f$ for any crossed module morphism f. The action $\mu^{H\otimes X}_{i,l}$ is the left action induced by H and $\mu^{H\otimes X}_{d,r}$ is the diagonal action according to (1.1). Analogously, in the dual manner the coactions $\nu^{H\otimes X}_{i,l}$ and $\nu^{H\otimes X}_{d,r}$ are defined. We formulate the equivalence theorem of crossed modules and Hopf bimodules in braided categories [BD1].

Theorem 1.4 *Let H be a Hopf algebra with isomorphic antipode in $\mathcal{C}$. Then the categories $\mathcal{DY}(\mathcal{C})^H_H$ and ${}^H_H\mathcal{C}^H_H$ are braided monoidal equivalent. The equivalence is given by $H \ltimes (\_) : \mathcal{DY}(\mathcal{C})^H_H \longrightarrow {}^H_H\mathcal{C}^H_H$ and ${}_H(\_) : {}^H_H\mathcal{C}^H_H \longrightarrow \mathcal{DY}(\mathcal{C})^H_H$.* ■

A tuple $(H, B, \underline{\eta}, \underline{\varepsilon})$ of bialgebras H and B, and of bialgebra morphisms $H \xrightarrow{\underline{\eta}} B \xrightarrow{\underline{\varepsilon}} H$ in $\mathcal{C}$ is called a bialgebra projection on H if $\underline{\varepsilon} \circ \underline{\eta} = \mathrm{id}_H$ [Rad]. Analogously Hopf algebra projections are defined. If $(H, B, \underline{\eta}, \underline{\varepsilon})$ is a bialgebra projection on H in $\mathcal{C}$, then $\underline{B} = (B, \mu_r^B, \mu_l^B, \nu_r^B, \nu_l^B)$ is an H-Hopf bimodule through the actions and coactions

$$\begin{aligned} \mu_l^B &= \mathrm{m}_B \circ (\underline{\eta} \otimes \mathrm{id}_B)\,, \quad \mu_r^B = \mathrm{m}_B \circ (\mathrm{id}_B \otimes \underline{\eta})\,, \\ \nu_l^B &= (\underline{\varepsilon} \otimes \mathrm{id}_B) \circ \Delta_B\,, \quad \nu_r^B = (\mathrm{id}_B \otimes \underline{\varepsilon}) \circ \Delta_B\,. \end{aligned} \tag{1.10}$$

For the formulation of the main theorem of this chapter the notation of a relative antipode of a Hopf bimodule $(X, \mu_r, \mu_l, \nu_r, \nu_l)$ is needed. It is the Hopf bimodule morphism $S_{X/H} : X \longrightarrow X$ defined by

$$S_{X/H} := M_X \circ (S \otimes \mathrm{id}_X \otimes S) \circ N_X \tag{1.11}$$

where $M_X := \mu_l \circ (\mathrm{id}_H \otimes \mu_r)$ and $N_X := (\mathrm{id}_H \otimes \nu_r) \circ \nu_l$. Every bialgebra projection $(H, B, \underline{\eta}_B, \underline{\varepsilon}_B)$ yields an object $\underline{B}$ according to (1.10) which is a bialgebra $(\underline{B}, \underline{\mathrm{m}}_B, \underline{\eta}_B, \underline{\Delta}_B, \underline{\varepsilon}_B)$ in ${}^H_H\mathcal{C}^H_H$. The multiplication $\underline{\mathrm{m}}_B$ and the comultiplication $\underline{\Delta}_B$ are defined uniquely through

$$\underline{\mathrm{m}}_B \circ (\mu_r^B \otimes \mathrm{id}_B) = \mathrm{m}_B \circ (\mathrm{id}_B \otimes {}^{{}^H_H\mathcal{C}^H_H}\lambda_B) \quad \text{and} \quad (\nu_r^B \otimes \mathrm{id}_B) \circ \underline{\Delta}_B = (\mathrm{id}_B \otimes ({}^{{}^H_H\mathcal{C}^H_H}\lambda_B)^{-1}) \circ \Delta_B\,. \tag{1.12}$$

Conversely every bialgebra $\underline{B} = (B, \underline{\mathrm{m}}_B, \underline{\eta}_B, \underline{\Delta}_B, \underline{\varepsilon}_B)$ in ${}^H_H\mathcal{C}^H_H$ yields the bialgebra $B = (B, \mathrm{m}_B, \eta_B, \Delta_B, \varepsilon_B)$ in $\mathcal{C}$ whose structure morphisms are given by

$$\mathrm{m}_B = \underline{\mathrm{m}}_B \circ \lambda^H_{B,B}\,, \quad \eta_B = \underline{\eta}_B \circ \eta_H\,, \quad \Delta_B = \rho^H_{B,B} \circ \underline{\Delta}_B\,, \quad \varepsilon_B = \varepsilon_H \circ \underline{\varepsilon}_B\,, \tag{1.13}$$

and furthermore $(H, B, \underline{\eta}_B, \underline{\varepsilon}_B)$ is a bialgebra projection on H. There also exists a correspondence of Hopf algebra structures. The antipode of $\underline{B}$ is given by $\underline{S}_B = M_B \circ (\mathrm{id}_H \otimes S_B \otimes \mathrm{id}_H) \circ N_B$ and for any Hopf algebra $\underline{B}$ in ${}^H_H\mathcal{C}^H_H$ the antipode of B is given by $S_B = \underline{S}_B \circ S_{B/H} = S_{B/H} \circ \underline{S}_B$.

Let us denote by H-Bialg-$\mathcal{C}$ the category of bialgebra projections on the Hopf algebra H. Its objects are the projections $B = \{B, \mathrm{m}_B, \eta_B, \Delta_B, \varepsilon_B, \underline{\eta}_B, \underline{\varepsilon}_B\}$ and its morphisms are bialgebra morphisms $f : B \longrightarrow D$ such that $f \circ \underline{\eta}_B = \underline{\eta}_D$ and $\underline{\varepsilon}_D \circ f = \underline{\varepsilon}_B$. Then the following central theorem can be formulated [BD1].

Theorem 1.5 *Let H be a Hopf algebra with isomorphic antipode in $\mathcal{C}$. Then the category H-*Bialg-$\mathcal{C}$ *of bialgebra projections on H and the category of H-Hopf bimodule bialgebras* Bialg-${}^H_H\mathcal{C}^H_H$ *are isomorphic. The isomorphism* Bialg-${}^H_H\mathcal{C}^H_H \underset{F}{\overset{G}{\rightleftarrows}} H$-Bialg-$\mathcal{C}$ *is given by (1.12) and (1.13) on the objects and through the identity mapping on the morphisms.* ■

Remark 2 Equation (1.6) for the braiding ${}^{{}^H_H\mathcal{C}^H_H}\Psi$ in ${}^H_H\mathcal{C}^H_H$ admits the following reformulation.

$$\begin{aligned} &\rho_{X' \otimes_H Y, X \otimes_H Y'} \circ (\mathrm{id}_{X'} \otimes_H {}^{{}^H_H\mathcal{C}^H_H}\Psi_{X,Y} \otimes_H \mathrm{id}_{Y'}) \circ \lambda_{X' \otimes_H X, Y \otimes_H Y'} \\ &= (\lambda_{X',Y} \otimes \lambda_{X,Y'}) \circ (\mathrm{id}_{X'} \otimes \Psi_{X,Y} \otimes \mathrm{id}_{Y'}) \circ (\rho_{X',X} \otimes \rho_{Y,Y'}) \end{aligned} \tag{1.14}$$

for Hopf bimodules X, Y, X', Y'. Then we can give another proof of the algebra morphism property of the comultiplication Δ_B in (1.13) (compare [BD1]). By definition (1.13) it holds $\Delta_B \circ \mathrm{m}_B = \rho_{B,B} \circ \underline{\Delta}_B \circ \underline{\mathrm{m}}_B \circ \lambda_{B,B}$. Using

(1.14) we derive

$$
\begin{aligned}
&(\mathrm{m}_B \otimes \mathrm{m}_B) \circ (\mathrm{id}_B \otimes \Psi_{B,B} \otimes \mathrm{id}_B) \circ (\Delta_B \otimes \Delta_B) \\
&= (\underline{\mathrm{m}}_B \otimes \underline{\mathrm{m}}_B) \circ (\rho_{B,B} \otimes \rho_{B,B}) \circ (\mathrm{id}_B \otimes \Psi_{B,B} \otimes \mathrm{id}_B) \circ (\lambda_{B,B} \otimes \lambda_{B,B}) \circ (\underline{\Delta}_B \otimes \underline{\Delta}_B) \\
&= (\underline{\mathrm{m}}_B \otimes \underline{\mathrm{m}}_B) \circ \rho_{B\otimes_H B, B\otimes_H B} \circ (\mathrm{id}_B \otimes_H {}^{H\mathcal{C}^H_H}_{H}\Psi_{B,B} \otimes_H \mathrm{id}_B) \circ \lambda_{B\otimes_H B, B\otimes_H B} \circ (\underline{\Delta}_B \otimes \underline{\Delta}_B) \\
&= \rho_{B,B} \circ (\underline{\mathrm{m}}_B \otimes_H \underline{\mathrm{m}}_B) \circ (\mathrm{id}_B \otimes_H {}^{H\mathcal{C}^H_H}_{H}\Psi_{B,B} \otimes_H \mathrm{id}_B) \circ (\underline{\Delta}_B \otimes_H \underline{\Delta}_B) \circ \lambda_{B,B} \\
&= \rho_{B,B} \circ \underline{\Delta}_B \circ \underline{\mathrm{m}}_B \circ \lambda_{B,B} = \Delta_B \circ \mathrm{m}_B \, .
\end{aligned}
\tag{1.15}
$$

Conversely one deduces from (1.15) the algebra morphism property of $\underline{\Delta}_B$ if it holds for Δ_B, since $\rho_{B,B}$ is monomorphism and $\lambda_{B,B}$ is epimorphism.

2 Factor Algebras and Ideals

In Chapter 2 we investigate factor (co-)algebras, bi- and Hopf algebras in abelian categories which arise from the canonical decomposition of (co-)algebra, bi- and Hopf algebra morphisms into monomorphism and epimorphism. Under more restrictive conditions to the category we will also consider (co-)ideals, bi- and Hopf ideals. The obtained results provide appropriate tools for the study of the braided exterior tensor algebras which will be constructed in Chapter 3. On the other hand these objects also play an important rôle in the examination of higher order bicovariant differential calculi in Chapter 4.

We suppose henceforth that the categories are (braided) monoidal abelian with bi-additive tensor product $\otimes$. Before we will fix additional conditions of the categories in the Definitions 2.3 and 2.5 we provide an appropriate definition of submodules generated by a given morphism.

Definition 2.1 *Let A be an algebra and (Y, μ_l) be an A-left module in the monoidal abelian category $\mathcal{C}$. For any morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ we define $A\langle f\rangle := \mathrm{Im}\,\big(\mu_l \circ (\mathrm{id}_A \otimes f)\big) = \mathrm{Ker}\,\mathrm{coker}\,\big(\mu_l \circ (\mathrm{id}_A \otimes f)\big)$. In an analogous manner $\langle f\rangle A$ is defined for an A-right module. If (Y, μ_l, μ_r) is an A-bimodule then $A\langle f\rangle A := \mathrm{Im}\,\big(\mu_r \circ (\mu_l \circ (\mathrm{id}_A \otimes f) \otimes \mathrm{id}_A)\big)$.*

Lemma 2.2 *Suppose that for any pair of epimorphisms g and h in $\mathcal{C}$ the tensor product $g \otimes h$ is again epimorphic. Then $B\langle f\rangle$, $\langle f\rangle B$ and $B\langle f\rangle B$ are B-left submodules, B-right submodules and B-sub-bimodules of Y respectively.*

Proof. We consider the B-left module Y and use the abbreviation $\phi := \mu_l \circ (id_B \otimes f)$. Then we obtain $\mathrm{coker}\,\phi \circ \mu_l \circ (\mathrm{id}_B \otimes \phi) = 0$. By assumption on the category $\mathcal{C}$ there exists a unique morphism $\mu'_l : B \otimes \mathrm{Im}\,\phi \longrightarrow \mathrm{Im}\,\phi$ such that $\mathrm{im}\,\phi \circ \mu'_l = \mu_l \circ (\mathrm{id}_B \otimes \mathrm{im}\,\phi)$. It follows

$$
\begin{gathered}
\mathrm{im}\,\phi \circ \mu'_l \circ (\mathrm{id}_B \otimes \mu'_l) = \mu_l \circ (\mathrm{m}_B \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{B\otimes B} \otimes \mathrm{im}\,\phi) = \mathrm{im}\,\phi \circ \mu'_l \circ (\mathrm{m}_B \otimes \mathrm{id}_{\mathrm{Im}\,\phi}) \, , \\
\mathrm{im}\,\phi \circ \mu'_l \circ (\eta_B \otimes \mathrm{id}_{\mathrm{Im}\,\phi}) = \mathrm{im}\,\phi \, .
\end{gathered}
\tag{2.1}
$$

This proves that $\big(B\langle f\rangle = \mathrm{Im}\,\phi, \mu'_l\big)$ is a B-left submodule of Y. Analogously the other cases of the proposition can be verified. ∎

The conditions on the category $\mathcal{C}$ used in Lemma 2.2 will be needed frequently in the following. We will therefore collect these properties in the next definition.

Definition 2.3 *The (braided) monoidal abelian category $\mathcal{C}$ is said to fulfill the $\otimes$-epimorphism property if the tensor product $f \otimes g$ of any two epimorphisms f and g in $\mathcal{C}$ is again epimorphic. The $\otimes$-monomorphism property is defined in the dual way. The category $\mathcal{C}$ is called $\otimes$-factor (braided) abelian if both the $\otimes$-epimorphism and the $\otimes$-monomorphism property hold in $\mathcal{C}$.*

For $\otimes$-factor (braided) abelian categories we provide some facts on factor (co-)algebras, factor bi- and Hopf algebras emerging out of the corresponding morphisms. This lemma will be used in the sequel, especially in Chapters 3 and 4.

Lemma 2.4 *Let $f : A \longrightarrow C$ be a (co-)algebra morphism in the $\otimes$-factor (braided) abelian category $\mathcal{C}$. Then the decomposition of f into its image and coimage $A \xrightarrow{\operatorname{coim} f} B \xrightarrow{\operatorname{im} f} C$ admits a unique (co-)algebra structure on B turning $\operatorname{im} f$ and $\operatorname{coim} f$ into (co-)algebra morphisms.*

The analogous results hold for bi- and Hopf algebras in $\otimes$-factor braided abelian categories.

Proof. The proof works similar as in classical ring theory. We will therefore only sketch it for the algebra case. Since $\mathcal{C}$ is a $\otimes$-factor abelian category there exists a unique morphism m_B making the following diagram commutative.

$$\begin{tikzcd} A \otimes A \arrow[r, "\operatorname{coim} f \otimes \operatorname{coim} f"] \arrow[d, "\mathrm{m}_A"'] & B \otimes B \arrow[r, "\operatorname{im} f \otimes \operatorname{im} f"] \arrow[d, "\mathrm{m}_B"] & C \otimes C \arrow[d, "\mathrm{m}_C"] \\ A \arrow[r, "\operatorname{coim} f"'] & B \arrow[r, "\operatorname{im} f"'] & C \end{tikzcd} \tag{2.2}$$

Then the definition $\eta_B := (\operatorname{coim} f) \circ \eta_A$ renders $(B, \mathrm{m}_B, \eta_B)$ a unital associative algebra. ∎

If not otherwise stated we will work in the subsequent chapters with $\otimes$-factor (braided) abelian categories. However, when we consider (co-)ideals, bi- and Hopf ideals, we will use categories which have a more restricted structure. This will be discussed in the next section.

Ideals

Definition 2.5 *We call the (braided) abelian category $\mathcal{C}$ $\otimes$-right-exact/$\otimes$-left-exact (braided) abelian if the following conditions are fulfilled. For any two epimorphisms/monomorphisms $f_i : X_i \longrightarrow Y_i$, $i \in \{1,2\}$ in $\mathcal{C}$ we suppose that the push-out/pull-back of the pair of morphisms $\{f_1 \otimes \mathrm{id}\,,\, \mathrm{id} \otimes f_2\}$ is given by $\{\mathrm{id} \otimes f_2\,,\, f_1 \otimes \mathrm{id}\}$. We say that the category $\mathcal{C}$ is $\otimes$-exact (braided) abelian if it is $\otimes$-left-exact abelian and $\otimes$-right-exact abelian.*

Both the dual categories of a $\otimes$-factor abelian and a $\otimes$-exact abelian category are again $\otimes$-factor abelian and $\otimes$-exact abelian respectively. In some cases we formulate statements which are not invariant under categorical duality. However, in this case the reader may immediately verify the dual assertion.

The $\otimes$-right-exact property means that for any pair of morphisms $l : X_1 \otimes Y_2 \longrightarrow Z$ and $h : Y_1 \otimes X_2 \longrightarrow Z$, which obey the identity $l \circ (\mathrm{id}_{X_1} \otimes f_2) = h \circ (f_1 \otimes \mathrm{id}_{X_2})$, there exists a unique morphism $k : Y_1 \otimes Y_2 \longrightarrow Z$ such that

$l = k \circ (f_1 \otimes \mathrm{id}_{Y_2})$ and $h = k \circ (\mathrm{id}_{Y_1} \otimes f_2)$. As a consequence $f \otimes g$ is epimorphic whenever f and g are epimorphic. Similarly in $\otimes$-left-exact abelian categories $k \otimes l$ is monomorphic when k and l are monomorphic. Hence $\otimes$-exact abelian categories are $\otimes$-factor abelian.

Suppose that $f_k : Y_k \longrightarrow Z$, $k \in \{1, 2\}$ are morphisms in an abelian category and the push-out of $\{\mathrm{coker}\, f_1, \mathrm{coker}\, f_2\}$ is given by $\{g_1, g_2\}$. Then

$$g_1 \circ f_1 = g_2 \circ f_2 = \mathrm{coker}\left(Y_1 \oplus Y_2 \xrightarrow{(f_1,f_2)} Z\right). \tag{2.3}$$

Of course, an analogous dual symmetric result holds. An immediate implication of this fact is the next corollary.

Corollary 2.6 *Let $\mathcal{C}$ be a $\otimes$-right-exact abelian category and X_1, X_2 be objects in $\mathcal{C}$ such that $X_1 \otimes \mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{C}} \otimes X_2$ are right exact. Then for any pair of morphisms $h_k : Y_k \longrightarrow X_k$, $k \in \{1, 2\}$ it holds*

$$\mathrm{coker}(h_1) \otimes \mathrm{coker}(h_2) = \mathrm{coker}\left((h_1 \otimes \mathrm{id}_{X_2}), (\mathrm{id}_{X_1} \otimes h_2)\right) \tag{2.4}$$

Proof. Applying equation (2.3) to $f_1 := h_1 \otimes \mathrm{id}_{X_2}$ and $f_2 := \mathrm{id}_{X_1} \otimes h_2$, and keeping in mind that $\mathcal{C}$ is $\otimes$-right-exact abelian and that $X_1 \otimes \mathrm{id}_{\mathcal{C}}$, $\mathrm{id}_{\mathcal{C}} \otimes X_2$ are right exact yields the result. ∎

Remark 3 Suppose that $X \otimes \mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{C}} \otimes X$ are right exact for every object X in the abelian monoidal category $\mathcal{C}$ – this holds if $\mathcal{C}$ is closed, for instance. Then the following statements are equivalent.

1. $\mathcal{C}$ is $\otimes$-right-exact abelian.
2. For any pair of epimorphisms $f_k : X_k \longrightarrow Y_k$, $k \in \{1, 2\}$ the tensor product $f_1 \otimes f_2$ is epimorphic. If $h : X_1 \otimes X_2 \longrightarrow Z$ factorizes over $f_1 \otimes \mathrm{id}_{X_2}$ and $\mathrm{id}_{X_1} \otimes f_2$, then h factorizes over $f_1 \otimes f_2$.
3. For any pair of morphisms h_1, h_2 the equation (2.4) holds.

Proof. Obviously (2) implies (1) which on the other hand yields (3) because of Corollary 2.6. It remains to show that (3) implies (1). For the epimorphisms f_k, $k \in \{1, 2\}$ the equation (2.4) yields the relation $f_1 \otimes f_2 = \mathrm{coker}\left((\ker f_1 \otimes \mathrm{id}_{X_2}, \mathrm{id}_{X_1} \otimes \ker f_2)\right)$. If the morphism h factorizes over $f_1 \otimes \mathrm{id}_{X_2}$ and $\mathrm{id}_{X_1} \otimes f_2$ then $h \circ (\ker f_1 \otimes \mathrm{id}_{X_2}) = 0$ and $h \circ (\mathrm{id}_{X_1} \otimes \ker f_2) = 0$. Hence h factorizes over $f_1 \otimes f_2$. ∎

We suppose in the remainder of this section that the categories under consideration are $\otimes$-exact (braided) abelian, although many results can be derived under weaker conditions.

In the following we imitate the well known definition of ideals, co-, bi- and Hopf ideals of classical ring theory.

Definition 2.7 *Let (A, m, η) be an algebra in $\mathcal{C}$. If there exists a subobject of A (represented by) $I \overset{\mathrm{i}}{\hookrightarrow} A$ such that*

$$\mathrm{im}\left(\mathrm{m} \circ \left((\mathrm{i} \otimes \mathrm{id}_A), (\mathrm{id}_A \otimes \mathrm{i})\right)\right) \subset \mathrm{i} \tag{2.5}$$

then (I, i) is called ideal in A. If (C, Δ, ε) is a coalgebra in $\mathcal{C}$ and $J \overset{\mathrm{j}}{\hookrightarrow} C$ is a subobject of C with

$$\mathrm{im}\,(\Delta \circ \mathrm{j}) \subset \mathrm{im}\left((\mathrm{j} \otimes \mathrm{id}_A), (\mathrm{id}_A \otimes \mathrm{j})\right), \quad \text{and} \quad \varepsilon \circ \mathrm{j} = 0 \tag{2.6}$$

then (J, j) is called coideal in C. For a bialgebra B in $\mathcal{C}$ a subobject (I, i) of B is called bi-ideal if both (I, i) is ideal and coideal in B; if B is a Hopf algebra then (I, i) is called Hopf ideal if in addition $\mathrm{coker}(\mathrm{i}) \circ S \circ \mathrm{i} = 0$.

Remark 4 The ideals of an algebra A in $\mathcal{C}$ are in one-to-one correspondence with the (equivalence classes of) subobjects of the regular bimodule A in ${}_A\mathcal{C}_A$.

Of course the definition of an ideal and of a coideal are not mutually dual in the categorical sense. For instance in the category of vector spaces the dual of a coideal is the quotient space of an algebra and a subalgebra together with the canonical surjective mapping.

The following propositions generalize classical results of ring theory to $\otimes$-exact braided abelian categories. Proposition 2.8 describes the correspondence of ideals and factor algebras. In Proposition 2.9 ideals generated by morphisms are described.

Proposition 2.8 *Let $\mathcal{C}$ be a $\otimes$-right-exact braided abelian category. Suppose that A is an algebra in $\mathcal{C}$ and $A \otimes \mathrm{id}_{\mathcal{C}}$ is right exact. Then every ideal of A represented by $I \overset{\mathrm{i}}{\hookrightarrow} A$ induces a unique algebra structure on the morphism* $\mathrm{coker}\,\mathrm{i} : A \longrightarrow \bar{A} := \mathrm{Coker}\,\mathrm{i}$. *Conversely every epimorphic algebra morphism $p : A \longrightarrow \bar{A}$ is isomorphic to the cokernel of a morphism* i *representing an ideal of A.*

Analogous relations hold in the case of coalgebras, bi- and Hopf algebras and their corresponding co-, bi- and Hopf ideals respectively.

Proof. If (A, m, η) is an algebra and $\mathrm{i} : I \hookrightarrow A$ is an ideal of A, then (2.5) holds, and we can apply Corollary 2.6 to obtain a unique morphism $\bar{\mathrm{m}} : \mathrm{Coker}\,\mathrm{i} \otimes \mathrm{Coker}\,\mathrm{i} \longrightarrow \mathrm{Coker}\,\mathrm{i}$ such that $\mathrm{coker}\,\mathrm{i} \circ \mathrm{m} = \bar{\mathrm{m}} \circ (\mathrm{coker}\,\mathrm{i} \otimes \mathrm{coker}\,\mathrm{i})$. Obviously $(\mathrm{Coker}\,\mathrm{i}, \bar{\mathrm{m}}, \bar{\eta} := \mathrm{coker}\,\mathrm{i} \circ \eta)$ is the unique algebra making $\mathrm{coker}\,\mathrm{i}$ an algebra morphism. Conversely, let $\mathrm{p} : A \longrightarrow \bar{A}$ be an epimorphic algebra morphism. Then for $\mathrm{i} := \ker \mathrm{p}$ the ideal property (2.5) is verified directly, $\mathrm{p} \circ \mathrm{m}_A \circ \big((\mathrm{i} \otimes \mathrm{id}_A)\,,\,(\mathrm{id}_A \otimes \mathrm{i})\big) = \mathrm{m}_P \circ (\mathrm{p} \otimes \mathrm{p}) \circ \big((\mathrm{i} \otimes \mathrm{id}_A)\,,\,(\mathrm{id}_A \otimes \mathrm{i})\big) = 0$.

Let (C, Δ, ε) be a coalgebra and $\mathrm{j} : J \longrightarrow C$ be a coideal of C. Then because of (2.6) there exists a unique morphism $\bar{\Delta} : \mathrm{Coker}\,\mathrm{j} \longrightarrow \mathrm{Coker}\,\mathrm{j} \otimes \mathrm{Coker}\,\mathrm{j}$ and a unique morphism $\bar{\varepsilon} : \mathrm{Coker}\,\mathrm{j} \longrightarrow \mathbf{1}$ such that $\bar{\Delta} \circ \mathrm{coker}\,\mathrm{j} = \Delta \circ (\mathrm{coker}\,\mathrm{j} \otimes \mathrm{coker}\,\mathrm{j})$ and $\bar{\varepsilon} \circ \mathrm{coker}\,\mathrm{j} = \varepsilon$. Similarly as in the algebra case it follows that $(\mathrm{Coker}\,\mathrm{j}, \bar{\Delta}, \bar{\varepsilon})$ is the unique coalgebra so that $\mathrm{coker}\,\mathrm{i}$ is a coalgebra morphism. Conversely an epimorphic coalgebra morphism $\mathrm{p} : C \longrightarrow \bar{C}$ leads to $(\mathrm{p} \otimes \mathrm{p}) \circ \Delta \circ \ker \mathrm{p} = 0$ and $\varepsilon \circ \ker \mathrm{p} = \bar{\varepsilon} \circ \mathrm{p} \circ \ker \mathrm{p} = 0$. Therefore (2.6) holds for $\mathrm{j} = \ker \mathrm{p}$.

Bi-ideals of a bialgebra B induce factor objects which are at the same time algebras and coalgebras. The bialgebra properties of B then yield the compatibility of both structures leading to the factor bialgebra. The existence of the antipode and the Hopf algebra relations of the factor bialgebra arising from a Hopf ideal of a Hopf algebra H can be proven similarly. ∎

Remark 5 1. In the coalgebra case of the previous proposition the $\otimes$-right-exactness of $\mathcal{C}$ and the right exactness of $C \otimes \mathrm{id}_{\mathcal{C}}$ need not be required.
2. Under the assumptions of Proposition 2.8 it follows from Proposition 2.4 that the kernel of any algebra morphism is an ideal whoose structure is compatible with the given algebra structure. Analogous statements are obtained for co-, bi- and Hopf algebras.

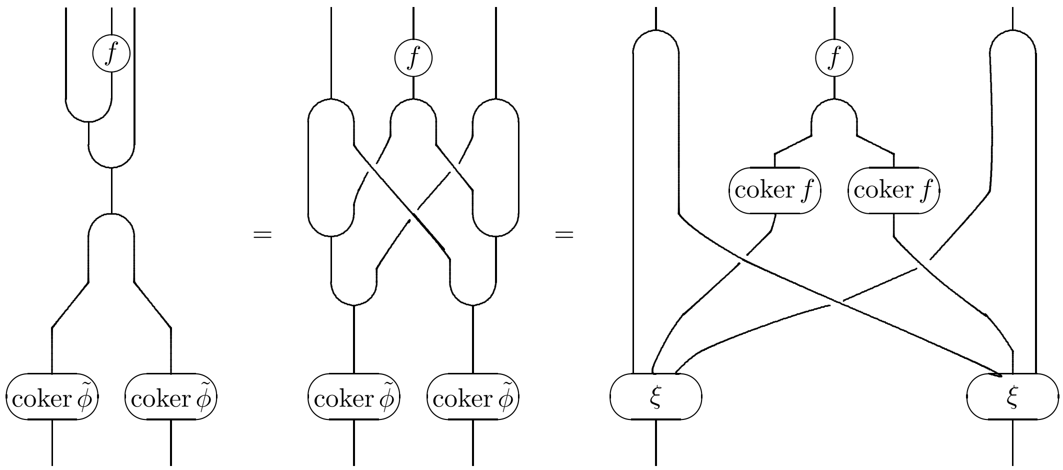

Figure 2: Proof of the coideal property of $\operatorname{im}\tilde\phi$.

Proposition 2.9 *Let A be an algebra and $g : X \longrightarrow A$ be any morphism in the category $\mathcal{C}$. Then $(g) := A\langle g\rangle A$ is an ideal in A. Suppose that B is a bialgebra (Hopf algebra) with right exact functor $B \otimes \mathrm{id}_{\mathcal{C}}$, and $f : X \to B$ is a morphism in $\mathcal{C}$ satisfying the condition*

$$(\operatorname{coker} f \otimes \operatorname{coker} f) \circ \Delta \circ f = 0\,, \quad \varepsilon_B \circ f = 0 \quad (\textit{and} \quad \operatorname{coker} f \circ S \circ f = 0)\,. \tag{2.7}$$

Then the ideal (f) generated by f is a bi-ideal (Hopf ideal) in B.

Proof. Given an algebra A, we consider the morphism $\tilde\phi_A := \mathrm{m}_A \circ (\mathrm{id}_A \otimes \mathrm{m}_A) \circ (\mathrm{id}_A \otimes g \otimes \mathrm{id}_A)$. Analogously the morphism $\tilde\phi_B$ is defined for a bialgebra B. Because of Lemma 2.2 we know that

$$\operatorname{coker}\tilde\phi \circ \mathrm{m} \circ \big((\operatorname{im}\tilde\phi \otimes \mathrm{id}), (\mathrm{id} \otimes \operatorname{im}\tilde\phi)\big) = \operatorname{coker}\tilde\phi \circ (\operatorname{im}\tilde\phi \circ \mu'_r, \operatorname{im}\tilde\phi \circ \mu'_l) = 0\,. \tag{2.8}$$

Thus (g) and (f) are ideals according to Definition 2.7. Since $B \otimes \mathrm{id}_{\mathcal{C}}$ is supposed to be right exact we obtain the following identities involving a unique morphism ξ.

$$\operatorname{coker}\tilde\phi_B \circ \mathrm{m}_B \circ (\mathrm{id}_B \otimes \mathrm{m}_B) = \xi \circ (\mathrm{id}_B \otimes \operatorname{coker} f \otimes \mathrm{id}_B) \tag{2.9}$$

where use has been made of the fact that $\tilde\phi_B$ factorizes over $(\mathrm{id}_B \otimes f \otimes \mathrm{id}_B)$. Keeping in mind that Δ_B is an algebra morphism we arrive at

$$(\operatorname{coker}\tilde\phi_B \otimes \operatorname{coker}\tilde\phi_B) \circ \Delta_B \circ \operatorname{im}\tilde\phi_B = 0\,. \tag{2.10}$$

For the derivation of (2.10) we used (2.9) and the assumption (2.7) for the morphism f. The explicit calculation yielding (2.10) is presented graphically in Figure 2. To verify the counit relation one observes that equivalent statements $\varepsilon_B \circ \operatorname{im}\tilde\phi_B = 0 \Leftrightarrow \varepsilon_B \circ \tilde\phi_B = 0 \Leftrightarrow \varepsilon_B \otimes \varepsilon_B \circ f \otimes \varepsilon_B = 0$ are fulfilled under the assumption (2.7). Using similar techniques as in the proof of (2.10) the antipode relation $\operatorname{coker}\tilde\phi_B \circ S_B \circ \operatorname{im}\tilde\phi_B = 0$ can be checked if B is a Hopf algebra. Therefore all the conditions of Definition 2.7 hold and the proposition is proved. ∎

3 Graded Categories, Complexes and Exterior Hopf Algebras

In this chapter we are considering graded categories and categories of complexes over braided abelian base categories $\mathcal{C}$. We are investigating (co-)algebras, bi- and Hopf algebras in these categories. In particular we study graded tensor algebras over an arbitrary object X in $\mathcal{C}$ and derive Hopf algebra structures on them using braided combinatorics. The corresponding braided antisymmetric tensor Hopf algebras and the braided exterior Hopf algebras of forms are deduced, and their universal properties are shown. In Chapter 4 these objects are shown to have a certain unique differential structure if X is a bicovariant differential calculus.
For the investigation of graded and complex categories we go some steps beyond and study functor categories $\mathcal{C}^{\mathcal{I}}$ over coalgebra categories $\mathcal{I}$. The notion of a coalgebra, bi- or Hopf algebra category is first given in a very general sense in [CF]. The properties of (quasitriangular) coalgebra categories allow us to define a (braided) monoidal structure on $\mathcal{C}^{\mathcal{I}}$ if $\mathcal{C}$ is (braided) monoidal. The reader who wants to avoid reading this rather formal investigation may skip the following section and should continue at the definition of graded categories and complexes over $\mathbb{N}_0$ or $\{0,1\}$ which are special examples of the general situation.

Matrix Categories, Coalgebra Categories and Functor Categories

Let $\Bbbk$ be a commutative, unital ring. In this section we suppose that the categories under consideration are $\Bbbk$-linear categories and the functors are $\Bbbk$-linear. In particular the ring $\Bbbk$ itself is a $\Bbbk$-linear category with a single object, $\mathrm{Ob}(\Bbbk) = \{\Bbbk\}$ and $\mathrm{End}_{\Bbbk}(\Bbbk) = \Bbbk$. For a given $\Bbbk$-linear category $\mathcal{C}$ we consider the matrix category $\mathrm{M}(\mathcal{C})$. It consists of objects $\vec{X} = (X_1, \ldots, X_n)$, $n \in \mathbb{N}_0$ which are finite ordered sets of objects $X_i \in \mathrm{Ob}(\mathcal{C})$, i $\in \{1, \ldots, n\}$. For $n = 0$ we formally denote by $\emptyset$ the "empty" zero object in $\mathrm{M}(\mathcal{C})$. The morphisms $\hat{g} : \vec{X} \longrightarrow \vec{Y}$ are given by $\hat{g} = \{g_{i\,j}\}_{(i,j)}$, where $g_{i\,j} : X_j \longrightarrow Y_i$ is a morphism in $\mathrm{Hom}_{\mathcal{C}}(X_j, Y_i)$. The composition of morphisms is matrix-like for every single component. Then obviously $\mathrm{M}(\mathcal{C})$ is also a $\Bbbk$-linear category.

A $\Bbbk$-linear functor $F : \mathcal{C} \longrightarrow \mathrm{M}(\mathcal{D})$ extends to a $\Bbbk$-linear functor $\hat{F} : \mathrm{M}(\mathcal{C}) \longrightarrow \mathrm{M}(\mathcal{D})$ through $\hat{F}(\vec{X}) = (F(X_1), \ldots, F(X_n))$ and $\hat{F}(\hat{g}) = \{F(g_{i\,j})\}$. Then the canonical embedding $\mathcal{C} \hookrightarrow \mathrm{M}(\mathcal{C})$ extends to the identity functor on $\mathrm{M}(\mathcal{C})$ and $(\hat{F} \circ G)^\wedge = \hat{F} \circ \hat{G}$. These facts permit the construction of the (large) category $\Bbbk$-Mcat. Its objects are (small) $\Bbbk$-linear categories, and morphisms $F \in \mathrm{Hom}_{\Bbbk\text{-Mcat}}(\mathcal{C}, \mathcal{D})$ are $\Bbbk$-linear functors $F : \mathcal{C} \longrightarrow \mathrm{M}(\mathcal{D})$.

For $\Bbbk$-Mcat a bi-functor $\underline{\otimes} : \Bbbk\text{-Mcat} \times \Bbbk\text{-Mcat} \longrightarrow \Bbbk\text{-Mcat}$ can be defined as follows. The object $\mathcal{C} \underline{\otimes} \mathcal{D}$ is the $\Bbbk$-linear category with objects $(X, Y) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$, and with morphisms $\mathrm{Hom}_{\mathcal{C} \underline{\otimes} \mathcal{D}}((X_1, Y_1), (X_2, Y_2)) = \mathrm{Hom}\, \mathcal{C}(X_1, X_2) \otimes_{\Bbbk} \mathrm{Hom}\, \mathcal{D}(Y_1, Y_2)$. For $\Bbbk$-linear functors $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ and $G : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ the $\Bbbk$-linear functor $F \underline{\otimes} G : \mathcal{C}_1 \underline{\otimes} \mathcal{D}_1 \longrightarrow \mathcal{C}_2 \underline{\otimes} \mathcal{D}_2$ is given by $F \underline{\otimes} G(X, Y) = (F(X), G(Y))$ in lexicographic ordering on the objects and by $F \underline{\otimes} G(f \otimes_{\Bbbk} g) = \{F(f)_{i\,j} \otimes_{\Bbbk} G(g)_{k\,l}\}$ on the morphisms. In addition functorial morphisms $\varphi_1 : F_1 \xrightarrow{\bullet} G_1$ and $\varphi_2 : F_2 \xrightarrow{\bullet} G_2$ compose to a functorial morphism $\varphi_1 \underline{\otimes} \varphi_2 : F_1 \underline{\otimes} G_1 \xrightarrow{\bullet} F_2 \underline{\otimes} G_2$ which is defined through $(\varphi_1 \underline{\otimes} \varphi_2)_{(X,Y)} = \{(\varphi_1)_{X,\,k\,l} \otimes_{\Bbbk} (\varphi_2)_{Y,\,m\,n}\}$.

Because of the symmetry of the tensor product in $\Bbbk$-mod it is possible to define an involutive functorial isomorphism $\underline{\tau} : \underline{\otimes} \overset{\bullet}{\cong} \underline{\otimes}^{\mathrm{op}}$ which is the analogue of the tensor transposition in $\Bbbk$-mod. It is given by $\underline{\tau}_{\mathcal{C}\,\mathcal{D}}(X, Y) = (Y, X)$ on the objects and by $\underline{\tau}_{\mathcal{C}\,\mathcal{D}}(f \otimes_{\Bbbk} g) = g \otimes_{\Bbbk} f$ on the (generating) morphisms. We collect these results in the following lemma.

Lemma 3.1 *The category* $(\Bbbk\text{-Mcat}, \underline{\otimes}, \Bbbk, \underline{\tau})$ *is symmetric monoidal. The functorial isomorphisms which are ruling the associativity and the unit object property are naturally induced by the monoidal structure of* $\Bbbk$-mod. ∎

Lemma 3.1 allows us to use the notion of a *coalgebra category* [CF] in $\Bbbk$-Mcat. It is a category $\mathcal{I}$ in $\Bbbk$-Mcat with $\Bbbk$-linear functors $\underline{\varepsilon_{\mathcal{I}}} : \mathcal{I} \longrightarrow \mathrm{M}(\Bbbk)$ and $\underline{\Delta_{\mathcal{I}}} : \mathcal{I} \longrightarrow \mathrm{M}(\mathcal{I} \,\underline{\otimes}\, \mathcal{I})$ such that $(\mathcal{I}, \underline{\Delta_{\mathcal{I}}}, \underline{\varepsilon_{\mathcal{I}}})$ is a coalgebra in $\Bbbk$-Mcat.

Since $\Bbbk$-Mcat is a 2-category where the functorial morphisms are the 2-morphisms, it is in particular possible to define the composition of functors and functorial morphisms [MP, Mac]. Let $F, G : \mathcal{C} \longrightarrow \mathrm{M}(\mathcal{D})$, $K : \mathcal{D} \longrightarrow \mathrm{M}(\mathcal{E})$ and $L : \mathcal{F} \longrightarrow \mathrm{M}(\mathcal{C})$ be functors in $\Bbbk$-Mcat. Given any functorial morphism $\varphi : F \overset{\bullet}{\longrightarrow} G$, then the compositions $K \diamond \varphi : K \circ F \overset{\bullet}{\longrightarrow} K \circ G$, with $(K \diamond \varphi)_X = K(\varphi_X)$ and $\varphi \diamond L : F \circ L \overset{\bullet}{\longrightarrow} G \circ L$, with $(\varphi \diamond L)_Y = \varphi_{L(Y)}$ are functorial morphisms. This will be used in the following definition.

Definition 3.2 *A coalgebra category $(\mathcal{I}, \underline{\Delta_{\mathcal{I}}}, \underline{\varepsilon_{\mathcal{I}}})$ is called quasitriangular if there exists a functorial isomorphism $\underline{R_{\mathcal{I}}} : \underline{\Delta_{\mathcal{I}}} \overset{\bullet}{\longrightarrow} \underline{\Delta_{\mathcal{I}}^{\mathrm{op}}} := \underline{\tau_{\mathcal{I},\mathcal{I}}} \circ \underline{\Delta_{\mathcal{I}}}$ satisfying the following dual hexagon identities.*

$$
\begin{aligned}
(\mathrm{id}_{\mathcal{I}} \,\underline{\otimes}\, \underline{\Delta_{\mathcal{I}}}) \diamond \underline{R_{\mathcal{I}}} &= \Big(\big((\underline{\tau_{\mathcal{I},\mathcal{I}}} \,\underline{\otimes}\, \mathrm{id}_{\mathcal{I}}) \diamond (\mathrm{id}_{\mathrm{id}_{\mathcal{I}}} \,\underline{\otimes}\, \underline{R_{\mathcal{I}}})\big) \circ (\underline{R_{\mathcal{I}}} \,\underline{\otimes}\, \mathrm{id}_{\mathrm{id}_{\mathcal{I}}})\Big) \diamond \underline{\Delta_{\mathcal{I}}}\,,\\
(\underline{\Delta_{\mathcal{I}}} \,\underline{\otimes}\, \mathrm{id}_{\mathcal{I}}) \diamond \underline{R_{\mathcal{I}}} &= \Big(\big((\mathrm{id}_{\mathcal{I}} \,\underline{\otimes}\, \underline{\tau_{\mathcal{I},\mathcal{I}}}) \diamond (\underline{R_{\mathcal{I}}} \,\underline{\otimes}\, \mathrm{id}_{\mathrm{id}_{\mathcal{I}}})\big) \circ (\mathrm{id}_{\mathrm{id}_{\mathcal{I}}} \,\underline{\otimes}\, \underline{R_{\mathcal{I}}})\Big) \diamond \underline{\Delta_{\mathcal{I}}}\,.
\end{aligned}
\tag{3.1}
$$

In (3.1) the composition "$\circ$" denotes the usual (pointwise) 2-composition of functorial morphisms.

Remark 6 One can write the identities (3.1) in a more common form.

$$
(\mathrm{id} \,\underline{\otimes}\, \underline{\Delta}) \diamond \underline{R} = (\underline{R}_{13} * \underline{R}_{12}) \qquad \text{and} \qquad (\underline{\Delta} \,\underline{\otimes}\, \mathrm{id}) \diamond \underline{R} = (\underline{R}_{13} * \underline{R}_{23})
$$

where obvious notations have been used.

Before we are going to formulate the proposition on braided monoidal structures of functor categories, we have to provide another two functors. For any $\Bbbk$-linear abelian monoidal category $\mathcal{C}$ (with $\Bbbk$-bilinear functor $\otimes$ henceforth) we define the $\Bbbk$-linear functor $\bullet_{\mathcal{C}} : \mathrm{M}(\mathcal{C} \,\underline{\otimes}\, \mathcal{C}) \longrightarrow \mathcal{C}$

$$
\bullet_{\mathcal{C}} : \left\{ \begin{array}{ccc} (X_i, Y_i)_{i=1}^n & \longmapsto & \bigoplus_{i=1}^n X_i \otimes Y_i \\ \Big\{ \sum_{k(j,i)} f_{j\,i}^k \otimes_{\Bbbk} g_{j\,i}^k \Big\}_{j,i} & \longmapsto & \Big(\sum_{k(j,i)} f_{j\,i}^k \otimes g_{j\,i}^k \Big)_{j,i} \end{array} \right. .
\tag{3.2}
$$

Similarly (using direct sums only) the functor $\times_{\mathcal{C}} : \mathrm{M}(\mathcal{C}) \longrightarrow \mathcal{C}$ may be defined.

Proposition 3.3 *Let $(\mathcal{I}, \underline{\Delta_{\mathcal{I}}}, \underline{\varepsilon_{\mathcal{I}}}, \underline{R_{\mathcal{I}}})$ be a quasitriangular coalgebra category and $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}}, \Psi^{\mathcal{C}})$ be a $\Bbbk$-linear braided abelian category. Then the category $(\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}, \otimes_{\mathcal{C}^{\mathcal{I}}}, \mathbf{1}_{\mathcal{C}^{\mathcal{I}}}, \Psi^{\mathcal{C}^{\mathcal{I}}})$ of $\Bbbk$-linear functors is again a $\Bbbk$-linear braided abelian category. The tensor product is given by $F \otimes_{\mathcal{C}^{\mathcal{I}}} G = \bullet_{\mathcal{C}} \circ (F \,\underline{\otimes}\, G) \circ \underline{\Delta_{\mathrm{I}}}$ on the functors and by $\varphi \otimes_{\mathcal{C}^{\mathcal{I}}} \chi = \bullet_{\mathcal{C}} \circ (\varphi \,\underline{\otimes}\, \chi) \circ \underline{\Delta_{\mathrm{I}}}$ on the corresponding functorial morphisms. The unit object is $\mathbf{1}_{\mathcal{C}^{\mathcal{I}}} = \eta_{\mathcal{C}} \circ \underline{\varepsilon_{\mathcal{I}}}$ with the functor $\eta_{\mathcal{C}} : \mathrm{M}(\Bbbk) \longrightarrow \mathcal{C}$, $\eta_{\mathcal{C}}(\Bbbk) = \mathbf{1}_{\mathcal{C}}$, $\eta_{\mathcal{C}}(\lambda) = \lambda \cdot \mathrm{id}_{\mathbf{1}_{\mathcal{C}}}$, and the braiding is defined by*

$$
\Psi^{\mathcal{C}^{\mathcal{I}}}_{F,G} = \big(\times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}} \diamond (F \,\underline{\otimes}\, G) \circ \underline{\Delta_{\mathcal{I}}^{\mathrm{op}}} \big) \circ \big(\bullet_{\mathcal{C}} \circ (F \,\underline{\otimes}\, G) \diamond \underline{R_{\mathcal{I}}} \big) : F \otimes_{\mathcal{C}^{\mathcal{I}}} G \longrightarrow G \otimes_{\mathcal{C}^{\mathcal{I}}} F\,.
\tag{3.3}
$$

The isomorphisms of associativity and of left and right unit multiplication are defined componentwise through the ones of $\mathcal{C}$. If $\mathcal{C}$ is in addition $\otimes$-factor/$\otimes$-exact abelian then $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$ is also $\otimes$-factor/$\otimes$-exact abelian.

Proof. Since functor categories are abelian if the codomaine category $\mathcal{C}$ is abelian (see e. g. [Mac]) one immediately verifies that $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$ is abelian. Straightforward computations prove the monoidal structure of $(\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}, \otimes_{\mathcal{C}^{\mathcal{I}}}, \mathbf{1}_{\mathcal{C}^{\mathcal{I}}})$, because $\mathcal{C}$ is monoidal. Without problems the functorial property $\Psi^{\mathcal{C}^{\mathcal{I}}} : \otimes_{\mathcal{C}^{\mathcal{I}}} \overset{\bullet}{\cong} \otimes^{\mathrm{op}}_{\mathcal{C}^{\mathcal{I}}}$ is proven. It remains to show the hexagon identities for the braiding $\Psi^{\mathcal{C}^{\mathcal{I}}}$. The use of the explicit form of the tensor product of $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$ leads (up to associativity) to

$$\begin{aligned}\Psi^{\mathcal{C}^{\mathcal{I}}}_{F,G\otimes_{\mathcal{C}^{\mathcal{I}}}K} = \big(\times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}} \diamond (\mathrm{id}_{\mathcal{C}} \underline{\otimes} \bullet_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \circ (\mathrm{id}_{\mathcal{I}} \underline{\otimes} \underline{\Delta}_{\mathcal{I}}) \circ \underline{\Delta}^{\mathrm{op}}_{\mathcal{I}}\big) \circ \\ \circ \big(\bullet_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}} \underline{\otimes} \bullet_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \circ (\mathrm{id}_{\mathcal{I}} \underline{\otimes} \underline{\Delta}_{\mathcal{I}}) \diamond \underline{R}_{\mathcal{I}}\big)\,.\end{aligned} \tag{3.4}$$

On the other hand a similar calculation yields

$$\begin{aligned}\Psi^{\mathcal{C}^{\mathcal{I}}}_{F,G} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathrm{id}_K = \big(\bullet_{\mathcal{C}} \diamond (\times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}} \underline{\otimes} \mathrm{id}_{\mathrm{id}_{\mathcal{C}}}) \diamond (F \underline{\otimes} G \underline{\otimes} K) \circ (\underline{\Delta}^{\mathrm{op}}_{\mathcal{I}} \underline{\otimes} \mathrm{id}_{\mathcal{I}}) \circ \underline{\Delta}_{\mathcal{I}}\big) \circ \\ \circ \big(\bullet_{\mathcal{C}} \circ (\bullet_{\mathcal{C}} \underline{\otimes} \mathrm{id}_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \diamond (\underline{R}_{\mathcal{I}} \underline{\otimes} \mathrm{id}_{\mathrm{id}_{\mathcal{I}}}) \circ \underline{\Delta}_{\mathcal{I}}\big)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\mathrm{id}_F \otimes_{\mathcal{C}^{\mathcal{I}}} \Psi^{\mathcal{C}^{\mathcal{I}}}_{G,K} \\ &= \big(\bullet_{\mathcal{C}} \diamond (\mathrm{id}_{\mathrm{id}_{\mathcal{C}}} \underline{\otimes} \times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}}) \diamond (\underline{\tau}_{\mathcal{C},\mathcal{C}} \underline{\otimes} \mathrm{id}_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \circ (\underline{\tau}_{\mathcal{I},\mathcal{I}} \underline{\otimes} \mathrm{id}_{\mathcal{I}}) \circ (\mathrm{id}_{\mathcal{I}} \underline{\otimes} \underline{\Delta}^{\mathrm{op}}_{\mathcal{I}}) \circ \underline{\Delta}_{\mathcal{I}}\big) \circ \\ &\circ \big(\bullet_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}} \underline{\otimes} \bullet_{\mathcal{C}}) \circ (\underline{\tau}_{\mathcal{C},\mathcal{C}} \underline{\otimes} \mathrm{id}_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \circ (\underline{\tau}_{\mathcal{I},\mathcal{I}} \underline{\otimes} \mathrm{id}_{\mathcal{I}}) \diamond (\mathrm{id}_{\mathrm{id}_{\mathcal{I}}} \underline{\otimes} \underline{R}_{\mathcal{I}}) \circ \underline{\Delta}_{\mathcal{I}}\big)\end{aligned} \tag{3.6}$$

where in equation (3.6) additionally the involutivity of $\underline{\tau}$ has been used. We compose (3.5) and (3.6) and obtain

$$\begin{aligned}&(\mathrm{id}_G \otimes_{\mathcal{C}^{\mathcal{I}}} \Psi^{\mathcal{C}^{\mathcal{I}}}_{F,K}) \circ (\Psi^{\mathcal{C}^{\mathcal{I}}}_{F,G} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathrm{id}_K) \\ &= \bullet_{\mathcal{C}} \diamond \Big(\big((\mathrm{id}_{\mathrm{id}_{\mathcal{C}}} \underline{\otimes} \times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}}) \diamond (\underline{\tau}_{\mathcal{C},\mathcal{C}} \underline{\otimes} \mathrm{id}_{\mathcal{C}})\big) \diamond (F \underline{\otimes} G \underline{\otimes} K) \circ (\mathrm{id}_{\mathcal{I}} \underline{\otimes} \underline{\Delta}_{\mathcal{I}}) \circ \underline{\Delta}^{\mathrm{op}}_{\mathcal{I}}\Big) \circ \\ &\quad \circ \big(\bullet_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}} \underline{\otimes} \bullet_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \circ (\mathrm{id}_{\mathcal{I}} \underline{\otimes} \underline{\Delta}_{\mathcal{I}}) \diamond \underline{R}_{\mathcal{I}}\big) \\ &= \big(\times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}} \diamond (\mathrm{id}_{\mathcal{C}} \underline{\otimes} \bullet_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \circ (\mathrm{id}_{\mathcal{I}} \underline{\otimes} \underline{\Delta}_{\mathcal{I}}) \circ \underline{\Delta}^{\mathrm{op}}_{\mathcal{I}}\big) \circ \\ &\quad \circ \big(\bullet_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}} \underline{\otimes} \bullet_{\mathcal{C}}) \circ (F \underline{\otimes} G \underline{\otimes} K) \circ (\mathrm{id}_{\mathcal{I}} \underline{\otimes} \underline{\Delta}_{\mathcal{I}}) \diamond \underline{R}_{\mathcal{I}}\big) \\ &= \Psi^{\mathcal{C}^{\mathcal{I}}}_{F,G\otimes_{\mathcal{C}^{\mathcal{I}}}K}\,.\end{aligned} \tag{3.7}$$

In the first equality of (3.7) we used the properties of $\underline{R}_{\mathcal{I}}$ according to Definition 3.2, the functoriality of $\Psi^{\mathcal{C}} : \otimes \overset{\bullet}{\cong} \otimes^{\mathrm{op}}$ and the involutivity of $\underline{\tau}$. The second equation follows because $\Psi^{\mathcal{C}}$ is a braiding and fulfills the hexagon identities. In a similar manner the second hexagon identity for $\Psi^{\mathcal{C}^{\mathcal{I}}}$ is verified.

$$\Psi^{\mathcal{C}^{\mathcal{I}}}_{F\otimes_{\mathcal{C}^{\mathcal{I}}}G,K} = (\Psi^{\mathcal{C}^{\mathcal{I}}}_{F,K} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathrm{id}_G) \circ (\mathrm{id}_F \otimes_{\mathcal{C}^{\mathcal{I}}} \Psi^{\mathcal{C}^{\mathcal{I}}}_{G,K})\,. \tag{3.8}$$

If $\mathcal{C}$ is $\otimes$-factor abelian and $\phi_j : F_j \overset{\bullet}{\longrightarrow} G_j$, $j \in \{1,2\}$ are two epimorphisms in $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$, then one verifies easily componentwise that $(\phi_1 \otimes_{\mathcal{C}^{\mathcal{I}}} \phi_2)$ is epimorphic in $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$. Similarly the $\otimes$-monomorphy property will be checked. Hence $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$ is $\otimes$-factor abelian.

Suppose that $\mathcal{C}$ is $\otimes$-exact abelian. Let again $\phi_j : F_j \overset{\bullet}{\longrightarrow} G_j$, $j \in \{1,2\}$ be two epimorphisms in $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$. For an object $\mathrm{i} \in \mathcal{I}$ denote $\underline{\Delta}_{\mathcal{I}}(\mathrm{i}) = \big((\mathrm{i}^{(1)}_1, \mathrm{i}^{(2)}_1), \ldots, (\mathrm{i}^{(1)}_n, \mathrm{i}^{(2)}_n)\big)$. If there are morphisms $\rho_1 : G_1 \otimes_{\mathcal{C}^{\mathcal{I}}} F_2 \overset{\bullet}{\longrightarrow} G_1 \otimes_{\mathcal{C}^{\mathcal{I}}} G_2$ and $\rho_2 : F_1 \otimes_{\mathcal{C}^{\mathcal{I}}} G_2 \overset{\bullet}{\longrightarrow} G_1 \otimes_{\mathcal{C}^{\mathcal{I}}} G_2$ such that $\rho_1 \circ (\phi_1 \otimes_{\mathcal{C}^{\mathcal{I}}} \mathrm{id}_{F_2}) = \rho_2 \circ (\mathrm{id}_{F_1} \otimes_{\mathcal{C}^{\mathcal{I}}} \phi_2)$, then it follows $\rho_1(\mathrm{i})_j \circ (\phi_1(\mathrm{i}^{(1)}_j) \otimes_{\mathcal{C}} \mathrm{id}_{F_2(\mathrm{i}^{(2)}_j)}) = \rho_2(\mathrm{i})_j \circ (\mathrm{id}_{F_1(\mathrm{i}^{(1)}_j)} \otimes_{\mathcal{C}} \phi_2(\mathrm{i}^{(2)}_j))$. There are unique morphisms $\rho(\mathrm{i})_j$ such that $\rho(\mathrm{i})_j \circ (\mathrm{id}_{G_1(\mathrm{i}^{(1)}_j)} \otimes_{\mathcal{C}} \phi_2(\mathrm{i}^{(2)}_j)) = \rho_1(\mathrm{i})_j$ and $\rho(\mathrm{i})_j \circ (\phi_1(\mathrm{i}^{(1)}_j) \otimes_{\mathcal{C}} \mathrm{id}_{G_2(\mathrm{i}^{(2)}_j)}) = \rho_2(\mathrm{i})_j$ because $\mathcal{C}$ is $\otimes$-exact abelian by assumption. From this fact it follows immediately that ρ is a natural morphism and that the $\otimes$-right-exact condition of Definition 2.5 is fulfilled for $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$. Analogously the $\otimes$-left-exact property can be shown. Hence $\Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$ is $\otimes$-exact abelian. ∎

Every coalgebra morphism in $\Bbbk$-Mcat induces an exact $\Bbbk$-linear monoidal functor on the corresponding functor categories. More precisely it holds

Corollary 3.4 *Let $\mathcal{I}$ and $\mathcal{J}$ be (quasitriangular) coalgebra categories and $\Theta : \mathcal{I} \longrightarrow \mathcal{J}$ be a coalgebra functor in $\Bbbk$-Mcat (such that $(\Theta \underline{\otimes} \Theta) \diamond \underline{R}_{\mathcal{I}} = \underline{R}_{\mathcal{J}} \diamond \Theta$). If $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}}, (\Psi^{\mathcal{C}}))$ is a $\Bbbk$-linear (braided) monoidal abelian category then an exact $\Bbbk$-linear (braided) monoidal functor $\mathcal{C}^{\Theta} : \Bbbk\text{-}\mathcal{C}^{\mathcal{J}} \longrightarrow \Bbbk\text{-}\mathcal{C}^{\mathcal{I}}$ can be defined by*

$$\mathcal{C}^{\Theta} \;:\; \begin{cases} F & \mapsto \quad \times_{\mathcal{C}} \circ F \circ \Theta \\ \varphi & \mapsto \quad \times_{\mathcal{C}} \diamond \varphi \diamond \Theta \end{cases} \tag{3.9}$$

where F is a functor in $\mathrm{Ob}(\Bbbk\text{-}\mathcal{C}^{\mathcal{J}})$ and φ is a functorial morphism in $\Bbbk\text{-}\mathcal{C}^{\mathcal{J}}$. If Θ is an equivalence of categories then $\mathcal{C}^{\Theta}$ is an equivalence.

Proof. One proves without difficulties that $\mathcal{C}^{\Theta}$ is a $\Bbbk$-linear functor. Since $\mathcal{C}^{\mathcal{I}}$ and $\mathcal{C}^{\mathcal{J}}$ are abelian the exactness of $\mathcal{C}^{\Theta}$ is shown on the several components of the natural morphisms. The coalgebra morphism properties $\underline{\Delta}_{\mathcal{J}} \circ \Theta = (\Theta \underline{\otimes} \Theta) \circ \underline{\Delta}_{\mathcal{I}}$ and $\underline{\varepsilon}_{\mathcal{J}} \circ \Theta = \underline{\varepsilon}_{\mathcal{I}}$ show immediately that $\mathcal{C}^{\Theta}$ is monoidal.

If $(\Theta \underline{\otimes} \Theta) \diamond \underline{R}_{\mathcal{I}} = \underline{R}_{\mathcal{J}} \diamond \Theta$ then the following equations hold.

$$\begin{aligned} \mathcal{C}^{\Theta}(\Psi^{\mathcal{C}^{\mathcal{J}}}_{F,G}) &= \times_{\mathcal{C}} \diamond \Big(\big(\times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}} \diamond (F \underline{\otimes} G) \circ \underline{\Delta}^{\mathrm{op}}_{\mathcal{J}} \big) \circ \big(\bullet_{\mathcal{C}} \circ (F \underline{\otimes} G) \diamond \underline{R}_{\mathcal{J}} \big) \Big) \\ &= \big(\times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}} \diamond (F \circ \Theta \underline{\otimes} G \circ \Theta) \circ \underline{\Delta}^{\mathrm{op}}_{\mathcal{I}} \big) \circ \big(\bullet_{\mathcal{C}} \circ (F \circ \Theta \underline{\otimes} G \circ \Theta) \diamond \underline{R}_{\mathcal{I}} \big) \\ &= \big(\times_{\mathcal{C}} \diamond \Psi^{\mathcal{C}} \diamond (\mathcal{C}^{\Theta}(F) \underline{\otimes} \mathcal{C}^{\Theta}(G)) \circ \underline{\Delta}^{\mathrm{op}}_{\mathcal{I}} \big) \circ \big(\bullet_{\mathcal{C}} \circ (\mathcal{C}^{\Theta}(F) \underline{\otimes} \mathcal{C}^{\Theta}(G)) \diamond \underline{R}_{\mathcal{I}} \big) \\ &= \Psi^{\mathcal{C}^{\mathcal{I}}}_{\mathcal{C}^{\Theta}(F), \mathcal{C}^{\Theta}(G)} . \end{aligned} \tag{3.10}$$

Thus the functor $\mathcal{C}^{\Theta}$ respects the braiding. ∎

The investigations of braided graded categories and of categories of complexes require special $\Bbbk$-linear categories $\underline{\mathbb{N}} \subset \underline{\mathbb{N}}_c$. Their objects are the natural numbers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and the morphisms are given by

$$\mathrm{Hom}_{\underline{\mathbb{N}}}(m,n) = \begin{cases} \Bbbk & \text{if } m = n, \\ \{0\} & \text{if } m \neq n, \end{cases} \quad \text{and} \quad \mathrm{Hom}_{\underline{\mathbb{N}}_c}(m,n) = \begin{cases} \Bbbk & \text{if } m = n, \\ \Bbbk \cdot \partial_m & \text{if } n = m+1, \\ \{0\} & \text{else.} \end{cases}$$

The compositions are defined by $\Bbbk$-linearity and the generating relations $\partial_{n+1} \cdot \partial_n = 0$ for all $n \in \mathbb{N}_0$ in $\underline{\mathbb{N}}_c$. The category $\underline{\mathbb{N}}$ is a quasitriangular coalgebra category through the $\Bbbk$-linear functors $\underline{\Delta}$ and $\underline{\varepsilon}$ defined on every object $n \in \mathrm{Ob}(\underline{\mathbb{N}})$ by

$$\underline{\Delta}(n) = ((0,n), (1, n-1), \dots, (n,0)) \quad \text{and} \quad \underline{\varepsilon}(n) = \begin{cases} \Bbbk \in \mathrm{Ob}(\mathrm{M}(\Bbbk)) & \text{if } n = 0 \\ \emptyset \in \mathrm{Ob}(\mathrm{M}(\Bbbk)) & \text{if } n \neq 0 \end{cases} \tag{3.11}$$

and by the obvious definition of $\underline{\Delta}$ and $\underline{\varepsilon}$ on the morphisms. Compatible quasitriangular structures $\underline{R}_n : \Delta(n) \xrightarrow{\bullet} \Delta^{\mathrm{op}}(n)$ on $\underline{\mathbb{N}}$ are characterized by invertible elements $\lambda \in \Bbbk^*$. The nonzero components of the corresponding morphisms $\underline{R}^{(\lambda)}_n$ are given by $(\underline{R}^{(\lambda)}_n)_{(\ell,k),(k,\ell)} = \lambda^{k\,\ell}$ for $k + \ell = n$. Hence $(\underline{\mathbb{N}}, \underline{\Delta}, \underline{\varepsilon}, \underline{R}^{(\lambda)})$ is a quasitriangular coalgebra category for any $\lambda \in \Bbbk^*$. The next lemma states that there are also quasitriangular structures on $\underline{\mathbb{N}}_c$.

Lemma 3.5 *The category* $(\underline{\mathbb{N}}_c, \underline{\Delta}_c, \underline{\varepsilon}_c, \underline{R}_c)$ *is a quasitriangular coalgebra category where the coalgebra structure on* $\underline{\mathbb{N}}_c$ *is given by* $\underline{\Delta}_{c|\underline{\mathbb{N}}} = \underline{\Delta}$, $\underline{\varepsilon}_{c|\underline{\mathbb{N}}} = \underline{\varepsilon}$ *and* $\underline{\Delta}_c(\partial_n)_{(r,s),(k,\ell)} = \delta_{(r,s),(k+1,\ell)}\, \partial_k \otimes_{\Bbbk} \mathrm{id}_\ell + \delta_{(r,s),(k,\ell+1)}\, (-1)^k \,\mathrm{id}_k \otimes_{\Bbbk} \partial_\ell$ *and* $\underline{\varepsilon}_c(\partial_n) = 0$. *The quasitriangular isomorphism* $\underline{R}_c$ *coincides with* $\underline{R}^{(-1)}$.

Proof. Obviously $\underline{\Delta}_c$ is a functor because of the definition of $\underline{\Delta}_c(\partial_n)$ which is adapted to the identity $\partial_{n+1} \circ \partial_n = 0$. For the prove of the functoriality of $\underline{R}^{(-1)}$ we note that the nonzero elements of both matrices $\underline{R}^{(-1)}_{n+1} \circ \Delta(\partial_n)$ and $\Delta^{\mathrm{op}}(\partial_n) \circ \underline{R}^{(-1)}_n$ coincide and are given by $(-1)^{(k+1)\ell} \cdot \partial_k \otimes \mathrm{id}_\ell$ and $(-1)^{k\ell} \cdot \mathrm{id}_k \otimes \partial_\ell$. ■

Remark 7 A more general form for $\underline{\Delta}_c(\partial_n)$ which does not necessarily induce a quasitriangular structure on $\underline{\mathbb{N}}_c$, is the morphism $\underline{\Delta}_c(\partial_n)_{(r,s),(k,\ell)} = \delta_{(r,s),(k+1,\ell)}\, \alpha_{k,\ell} \cdot \partial_k \otimes_{\Bbbk} \mathrm{id}_\ell + \delta_{(r,s),(k,\ell+1)}\, (-1)^k \,\beta_{k,\ell} \cdot \mathrm{id}_k \otimes_{\Bbbk} \partial_\ell$ where $\alpha_{k,\ell}, \beta_{k,\ell} \in \Bbbk$. The coassociativity of $\underline{\Delta}_c$ and the identity $\partial_{n+1} \circ \partial_n = 0$ are equivalent to $\alpha_{k,\ell} = \alpha_{k+1} \cdot \alpha_{k+2} \cdot \ldots \cdot \alpha_{k+\ell}$ and $\beta_{k,\ell} = \beta_{\ell+1} \cdot \beta_{\ell+2} \cdot \ldots \cdot \beta_{\ell+k}$ and $\alpha_{k,\ell} \cdot \beta_{k+1,\ell} + \beta_{k,\ell} \cdot \alpha_{k,\ell+1} = \alpha_{k+1} \cdot \ldots \cdot \alpha_{k+\ell} \cdot \beta_{\ell+1} \cdot \ldots \cdot \beta_{\ell+k} \cdot (\alpha_{k+\ell+1} + \beta_{k+\ell+1}) = 0$, for any α_i, $\beta_i \in \Bbbk$. In the case where all α_i, β_i are invertible, these conditions imply that this coalgebra structure is equivalent to the one described in Lemma 3.5. There also exist other solutions for $\underline{\Delta}_c$ which are deviating from the structure of $\underline{\Delta}_c$ in Lemma 3.5. For example the elements $\alpha_1 = \beta_1 = 0$, $\alpha_2 = 1$, β_2 arbitrary, and $\alpha_n = -\beta_n = 1$ for $n \geq 3$ fulfill the conditions.

Graded Categories and Complexes

We return to the initial conditions fixed in Chapter 2 and direct our attention therefore to ⊗-factor braided abelian categories $\mathcal{C}$ and to the commutative ring $\Bbbk := \mathrm{End}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}})$. The full subcategories $\mathbf{2}$ and $\mathbf{2}_c$ of $\underline{\mathbb{N}}$ and $\underline{\mathbb{N}}_c$ respectively, with objects $\{0, 1\}$ are quasitriangular colagebra subcategories. Denote by I either the discrete category $\mathrm{I} = \mathbf{2}$ or $\mathrm{I} = \underline{\mathbb{N}}$ and by I_c the categories $\mathrm{I}_c = \mathbf{2}_c$ or $\mathrm{I}_c = \underline{\mathbb{N}}_c$. The I-graded category $\mathcal{C}^{\mathrm{I}}$ over $\mathcal{C}$ and the category $\mathcal{C}^{\mathrm{I}_c}$ of I-graded complexes are the functor categories which we are interested in in the sequel (see e. g. [Hus, Mac]). The categories $\mathcal{C}^{\mathrm{I}}$, $\mathcal{C}^{\mathrm{I}_c}$ are ⊗-factor braided abelian, because of Proposition 3.3. For completeness we will give below the explicit braided monoidal structure of these categories.

1. The objects of the graded category $\mathcal{C}^{\mathrm{I}}$ over $\mathrm{I} = \mathbf{2}$ or $\mathrm{I} = \underline{\mathbb{N}}$ are given by $\hat{X} = (X_0, X_1, \ldots)$ where $X_j \in \mathrm{Ob}(\mathcal{C})$ for all $j \in \mathrm{I}$. The morphisms of $\mathcal{C}^{\mathrm{I}}$ are of the form $\hat{f} = (f_0, f_1, \ldots) : \hat{X} \longrightarrow \hat{Y}$ where $f_j : X_j \longrightarrow Y_j$ is a morphism in $\mathcal{C}$ for all $j \in \mathrm{I}$.

2. The category $\mathcal{C}^{\mathrm{I}_c}$ of I-graded complexes consists of objects $(\hat{X}, \hat{\mathrm{d}})$ where $\hat{X} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{I}})$, and $\hat{\mathrm{d}} = (\mathrm{d}_0, \mathrm{d}_1, \ldots)$ is a differential such that for the morphisms $\mathrm{d}_j : X_j \longrightarrow X_{j+1}$ in $\mathcal{C}$ it holds $\mathrm{d}_{j+1} \circ \mathrm{d}_j = 0$ for all $j \in \mathrm{I}$ if $\mathrm{I} = \underline{\mathbb{N}}$. The morphisms $\hat{f} : (\hat{X}, \hat{\mathrm{d}}) \longrightarrow (\hat{Y}, \hat{\mathrm{d}}')$ in $\mathcal{C}^{\mathrm{I}_c}$ are morphisms in $\mathcal{C}^{\mathrm{I}}$ which obey the relations $f_{j+1} \circ \mathrm{d}_j = \mathrm{d}'_j \circ f_j$ $\forall\ j, j+1 \in \mathrm{I}$.

3. The unit objects of $\mathcal{C}^{\mathrm{I}}$ and $\mathcal{C}^{\mathrm{I}_c}$ are both given by $\hat{\mathbf{1}} = (\mathbf{1}_{\mathcal{C}}, 0, 0, \ldots)$. The tensor product is defined through $(\hat{X} \otimes \hat{Y})_n = \bigoplus_{k+l=n} X_k \otimes Y_l$ where $n, k, l \in \mathrm{I}$, and the tensor product of I-graded and complex morphisms is built analogously. The differential of a tensor product in $\mathcal{C}^{\mathrm{I}_c}$ is of the form $(\hat{\mathrm{d}}_{\hat{X} \otimes \hat{Y}})_n = \sum_{k+l=n} [\mathrm{d}_{\hat{X},k} \otimes \mathrm{id}_{Y_l} + (-1)^k \,\mathrm{id}_{X_k} \otimes \mathrm{d}_{\hat{Y},l}]$. The category $\mathcal{C}^{\mathrm{I}}$ admits a family of braidings given by $(\hat{\Psi}^{(\lambda)}_{\hat{X},\hat{Y}})_n = \bigoplus_{k+l=n} \lambda^{k\,l} \Psi_{X_k, Y_l}$ for any $\lambda \in \mathrm{Aut}_{\,\mathcal{C}}(\mathbf{1}_{\mathcal{C}})$[1]. The braiding in $\mathcal{C}^{\mathrm{I}_c}$ is given by $(\hat{\Psi}^{(-1)}_{\hat{X},\hat{Y}})$ forced by the nilpotency of the differentials $\hat{\mathrm{d}}$.

[1]The braidings in $\mathcal{C}^{\mathrm{I}}$ and $\mathcal{C}^{\mathrm{I}_c}$ are bi-additive, $\hat{\Psi}^{(\lambda)}_{\hat{X} \oplus \hat{Y}, \hat{Z}} = \hat{\Psi}^{(\lambda)}_{\hat{X}, \hat{Z}} \oplus \hat{\Psi}^{(\lambda)}_{\hat{Y}, \hat{Z}}$, because the braiding in $\mathcal{C}$ is bi-additive.

Consider the category $\underline{\mathbb{N}}$ with the quasitriangular structure $\underline{R}^{(-1)}$. Then there are coalgebra category morphisms $\Bbbk \leftrightarrows \underline{\mathbb{N}} \leftrightarrows \underline{\mathbb{N}}_c$ which are compatible with $\underline{R}^{(-1)}$. The right arrows are natural inclusions where $\Bbbk$ is identified with $\mathrm{End}(0)$. The functor $\underline{\mathbb{N}} \longrightarrow \Bbbk$ is the counit, and $\underline{\mathbb{N}}_c \longrightarrow \underline{\mathbb{N}}$ is identical on objects and sends ∂_n to 0. As an implication of Corollary 3.4 the following exact braided monoidal functors between $\mathcal{C}$, $\mathcal{C}^{\mathrm{I}}$ and $\mathcal{C}^{\mathrm{I}_c}$ can be deduced.

$$\mathcal{C} \leftrightarrows \mathcal{C}^{\mathrm{I}} \leftrightarrows \mathcal{C}^{\mathrm{I}_c} \tag{3.12}$$

The right arrows are the well known canonical inclusions given by the assignments $X \mapsto \hat{X} := (X, 0, 0, \dots)$ for all $X \in \mathrm{Ob}(\mathcal{C})$, and $\hat{Y} \mapsto (\hat{Y}, \hat{\mathrm{d}} := 0)$ for all $\hat{Y} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{I}})$. Analogously the assignments for the morphisms will be defined. In the sequel we will use this identification for embedding the objects and morphisms of the several categories into the graded category or the category of complexes respectively.

Remark 8 We have to give a comment on the usual notation of matrix elements of graded morphisms which we use subsequently. If $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ is a morphism in $\mathcal{C}^{\mathrm{I}}$ and for all $n \in \mathrm{I}$ the components $X_n = \bigoplus_{k+l=n} U_k^{(n)} \otimes V_l^{(n)}$ and $Y_n = \bigoplus_{k+l=n} W_k^{(n)} \otimes Z_l^{(n)}$ are direct sums of tensor products we denote by

$$f_{r,s;k,l} : U_k^{(n)} \otimes V_l^{(n)} \longrightarrow W_r^{(n)} \otimes Z_s^{(n)} \tag{3.13}$$

the corresponding matrix element of $\hat{f}$. If it is clear from the context we denote by $f_{m,n}$ either the morphism

$$f_{m,n} := \begin{pmatrix} f_{0,m+n;m,n} \\ \vdots \\ f_{m+n,0;m,n} \end{pmatrix} : U_m^{m+n} \otimes V_n^{m+n} \longrightarrow Y_{m+n} \tag{3.14}$$

or the morphism

$$f_{m,n} := \left(f_{m,n;0,m+n}, \dots , f_{m,n;m+n,0}\right) : X_{m+n} \longrightarrow W_m^{m+n} \otimes Z_n^{m+n} \tag{3.15}$$

Of course, the same notation is used if only $\hat{X}$ or $\hat{Y}$ consists of components which are direct sums of tensor products. One proceeds analogously if there are more than two tensor factors.

In the following we are investigating algebraic structures on objects in graded or complex categories. Similar considerations as in [Mal, Man, Wor] immediately lead to

Proposition 3.6 *Let $\hat{X}$ be an object in the $\otimes$-factor braided abelian category $\mathcal{C}^{\mathrm{I}}$ where $\mathrm{I} = \mathbf{2}$ or $\mathrm{I} = \underline{\mathbb{N}}$. Then it holds in particular*

1. *If $\hat{X}$ is an algebra in $\mathcal{C}^{\mathrm{I}}$ then X_0 is an algebra in $\mathcal{C}$ and X_n is an X_0-bimodule for any $n \in \mathrm{I}$.*
2. *If $\hat{X}$ is a coalgebra in $\mathcal{C}^{\mathrm{I}}$ then X_0 is a coalgebra in $\mathcal{C}$ and X_n is an X_0-bicomodule for any $n \in \mathrm{I}$.*
3. *If $\hat{X}$ is a bi- or Hopf algebra in $\mathcal{C}^{\mathrm{I}}$ then X_0 is a bi- or Hopf algebra in $\mathcal{C}$ respectively and X_n is an X_0-Hopf bimodule for any $n \in \mathrm{I}$.*

Proof. Suppose that $(\hat{X}, \hat{\mathrm{m}}, \hat{\eta})$ is an algebra in $\mathcal{C}^{\mathrm{I}}$ then $\hat{\mathrm{m}} \circ (\mathrm{id}_{\hat{X}} \otimes \hat{\mathrm{m}}) = \hat{\mathrm{m}} \circ (\hat{\mathrm{m}} \otimes \mathrm{id}_{\hat{X}})$. Looking in particular at the matrix components $(0,0,0)$ and $(0,0,n)$, $(n,0,0)$, $(0,n,0)$ of this equation, it follows that $(X_0, \mathrm{m}_{0,0})$ is an algebra and $(X_n, \mathrm{m}_{n,0}, \mathrm{m}_{0,n})$ is an X_0-bimodule respectively. The unital property of X_0 and X_n is given by the 0th and nth component of the equations $\hat{\mathrm{m}} \circ (\mathrm{id}_{\hat{X}} \otimes \hat{\eta}) = \mathrm{id}_{\hat{X}} = \hat{\mathrm{m}} \circ (\hat{\eta} \otimes \mathrm{id}_{\hat{X}})$ respectively. The dual statement for coalgebras is obtained analogously. In the case that $\hat{X}$ is a bialgebra the additional relation $\hat{\Delta} \circ \hat{\mathrm{m}} = (\hat{\mathrm{m}} \otimes \hat{\mathrm{m}}) \circ (\hat{\mathrm{id}} \otimes \hat{\Psi}^{(\lambda)} \otimes \hat{\mathrm{id}}) \circ (\hat{\Delta} \otimes \hat{\Delta})$ leads to the third statement of the proposition. ■

Remark 9 If $\mathrm{I} = \mathbf{2}$ the particular statements (1), (2) and (3) in Proposition 3.6 are equivalences.

Remark 10 For complexes we obtain analogous statements as in Propositon 3.6. The differential $\hat{\mathrm{d}}$ of a coalgebra $(\hat{X}, \hat{\mathrm{d}}) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{I}_c})$ is an X_0-bicomodule morphisms. If $(\hat{X}, \hat{\mathrm{d}})$ is an algebra in $\mathcal{C}^{\mathrm{I}_c}$ then the differential $\hat{\mathrm{d}}$ obeys a kind of Leibniz rule since the multiplication of $(\hat{X}, \hat{\mathrm{d}})$ is a complex morphism by assumption. In particular it holds $\mathrm{d}_0 \circ \mathrm{m}_{X_0} = \mu_r^{X_1} \circ (\mathrm{d}_0 \otimes \mathrm{id}_{X_0}) + \mu_l^{X_1} \circ (\mathrm{id}_{X_0} \otimes \mathrm{d}_0)$, where $\mu_r^{X_1}$ and $\mu_l^{X_1}$ are the right and left action of X_0 on X_1 respectively.

Remark 11 If $\hat{X}$ is a Hopf algebra in $\mathcal{C}^{\mathrm{I}}$ and X_0 is flat with isomorphic antipode in $\mathcal{C}$ then the canonical morphisms $X_0 \longrightarrow \hat{X} \longrightarrow X_0$ form a Hopf algebra projection on X_0 in $\mathcal{C}^{\mathrm{I}}$. Hence $(\underline{\hat{X}}, \underline{\mathrm{m}}_{\hat{X}}, \underline{\eta}_{\hat{X}}, \underline{\Delta}_{\hat{X}}, \underline{\varepsilon}_{\hat{X}}, \underline{S}_{\hat{X}})$ according to Theorem 1.5 is a Hopf algebra in ${}^{X_0}_{X_0}(\mathcal{C}^{\mathrm{I}})^{X_0}_{X_0}$.

Using Proposition 3.6 we derive necessary and sufficient criteria for a graded bialgebra to be a Hopf algebra. This is a generalization of the ideas of [Ma5]. The antipode of the form (3.16) has been used implicitely in [Brz] in the special case of Woronowicz's exterior algebra $\Gamma^{\wedge}$.

Proposition 3.7 *Let $(\hat{B}, \hat{\mathrm{m}}, \hat{\Delta})$ be a bialgebra in $\mathcal{C}^{\mathrm{I}}$ or in $\mathcal{C}^{\mathrm{I}_c}$. Then $\hat{B}$ is a Hopf algebra if and only if B_0 (with the bialgebra structure morphisms induced from $\hat{B}$) is a Hopf algebra in $\mathcal{C}$. In this case the antipode of $\hat{B}$ is successively given by*

$$S_n = -\sum_{k=1}^{n} \left(\mathrm{m}^{(2)}_{0,k,n-k} \circ (S_0 \otimes \mathrm{id}_{B_k} \otimes S_{n-k}) \circ \Delta^{(2)}_{0,k,n-k} \right) \tag{3.16}$$

for all $n \in \mathrm{I}$, where $\hat{\mathrm{m}}^{(2)} = \hat{\mathrm{m}} \circ (\hat{\mathrm{id}} \otimes \hat{\mathrm{m}})$ and $\hat{\Delta}^{(2)} = (\hat{\mathrm{id}} \otimes \hat{\Delta}) \circ \hat{\Delta}$.

Proof. The sufficiency has been proved in Proposition 3.6.3. Suppose that B_0 is a Hopf algebra in $\mathcal{C}$ and $\hat{B}$ is a bialgebra in $\mathcal{C}^{\mathrm{I}}$. We investigate the morphism $\hat{S}$ in (3.16) and show that it is an antipode of $\hat{B}$. By assumption it holds $\mathrm{m}_{0,0} \circ (\mathrm{id}_{B_0} \otimes S_0) \circ \Delta_{0,0} = \eta_0 \circ \varepsilon_0 = \mathrm{m}_{0,0} \circ (S_0 \otimes \mathrm{id}_{B_0}) \circ \Delta_{0,0}$. For $n > 0$ we obtain the following equations

with $\hat{\mathrm{m}}^{(3)} := \hat{\mathrm{m}} \circ (\hat{\mathrm{id}}_{\hat{B}} \otimes \hat{\mathrm{m}}) \circ (\hat{\mathrm{id}}_{\hat{B}} \otimes \hat{\mathrm{id}}_{\hat{B}} \otimes \hat{m})$, and $\hat{\Delta}^{(3)}$ defined in the dual analogous way.

$$
\begin{aligned}
\big(\hat{m} \circ (\hat{\mathrm{id}}_{\hat{B}} \otimes \hat{S}) \circ \hat{\Delta}\big)_n &= \mathrm{m}_{0,n} \circ (\mathrm{id}_{B_0} \otimes S_n) \circ \Delta_{0,n} + \sum_{k=1}^{n} \mathrm{m}_{k,n-k} \circ (\mathrm{id}_{B_k} \otimes S_{n-k}) \circ \Delta_{k,n-k} \\
&= \sum_{k=1}^{n} \Big(- \mathrm{m}^{(3)}_{0,0,k,n-k} \circ (\mathrm{id}_{B_0} \otimes S_0 \otimes \mathrm{id}_{B_k} \otimes S_{n-k}) \circ \Delta^{(3)}_{0,0,k,n-k} + \\
&\quad + \mathrm{m}_{k,n-k} \circ (\mathrm{id}_{B_k} \otimes S_{n-k}) \circ \Delta_{k,n-k} \Big) \\
&= \sum_{k=1}^{n} (-S_{n-k} + S_{n-k}) = 0
\end{aligned}
\tag{3.17}
$$

where we used (3.16) in the second equation for S_n and the (co-)associativity and the antipode property of S_0 in the third equation. Thus we obtain $\hat{\mathrm{m}} \circ (\hat{\mathrm{id}}_{\hat{B}} \otimes \hat{S}) \circ \hat{\Delta} = \hat{\eta} \circ \hat{\varepsilon}$. Similarly $\hat{\mathrm{m}} \circ (\hat{S} \otimes \hat{\mathrm{id}}_{\hat{B}}) \circ \hat{\Delta} = \hat{\eta} \circ \hat{\varepsilon}$ can be proven.

If $(\hat{B}, \hat{\mathrm{d}})$ is a bialgebra in $\mathcal{C}^{\mathrm{I}_c}$ and B_0 is a Hopf algebra in $\mathcal{C}$ it follows that $\hat{B}$ is a Hopf algebra in $\mathcal{C}^{\mathrm{I}}$ according to the previous considerations. Then

$$
\begin{aligned}
\hat{S} \circ \hat{\mathrm{d}} &= \hat{\mathrm{m}} \circ (\hat{S} \otimes \hat{\eta} \circ \hat{\varepsilon}) \circ \hat{\Delta} \circ \hat{\mathrm{d}} \\
&= \hat{\mathrm{m}} \circ (\hat{S} \otimes \hat{\mathrm{id}}) \circ \hat{\mathrm{d}}^{\otimes} \circ (\hat{\mathrm{id}} \otimes \hat{\eta} \circ \hat{\varepsilon}) \circ \hat{\Delta} \\
&= \hat{\mathrm{m}} \circ (\hat{S} \otimes \hat{\mathrm{id}}) \circ (\mathrm{id} \otimes \hat{\mathrm{m}}) \circ (\hat{\Delta} \otimes \hat{\mathrm{id}}) \circ \hat{\mathrm{d}}^{\otimes} \circ (\hat{\mathrm{id}} \otimes \hat{S}) \circ \hat{\Delta} \\
&= (\hat{\varepsilon} \otimes \hat{\mathrm{id}}) \circ \hat{\mathrm{d}}^{\otimes} \circ (\hat{\mathrm{id}} \otimes \hat{S}) \circ \hat{\Delta} \\
&= \hat{\mathrm{d}} \circ \hat{S}\,.
\end{aligned}
\tag{3.18}
$$

In the second, the third and the fifth equation of (3.18) we used that $(\hat{B}, \hat{\mathrm{d}})$ is a bialgebra in $\mathcal{C}^{\mathrm{I}_c}$. ■

Remark 12 In (3.18) we only have to require that $(\hat{B}, \hat{\mathrm{d}})$ is an algebra and a coalgebra in $\mathcal{C}^{\mathrm{I}_c}$. The techniques we applied in (3.18) can also be used in particular to show that any Hopf algebra which is a (co-)module bialgebra is automatically a (co-)module Hopf algebra.

Exterior Hopf Algebras

Next we consider the tensor algebra generated by an object $X \in \mathrm{Ob}(\mathcal{C})$. It is commonly defined as the graded object $T_{\mathcal{C}}(X)$ in $\mathcal{C}^{\underline{\mathbb{N}}}$ with $(T_{\mathcal{C}}(X))_0 := \mathbf{1}_{\mathcal{C}}$ and $(T_{\mathcal{C}}(X))_j := X^{\otimes j}$ the j-fold tensor product of X for $j \geq 1$. Before we construct Hopf algebra structures on $T_{\mathcal{C}}(X)$ we outline braided combinatorics which generalizes the usual symmetric combinatorics. We extend the ideas of [Ma4, Wor]. Since $(\mathcal{C}, \mathbf{1}, \otimes, \Psi)$ is a braided category it is possible to define a mapping of the symmetric group S_j into representations of the braid group B_j [Art] on the object $X^{\otimes j}$ for any $j \in \mathbb{N}_0$. For that we look at the so-called reduced expression. A reduced expression of a permutation $\sigma \in S_j$ is a product decomposition of σ into a minimal number of next neighbour transpositions t_a, $a \in \{1, \ldots, j-1\}$ which permute a with $a+1$. Two such decompositions of σ can be transformed into each other by applying braid group relations only [Wor]. The involutivity $t_a^2 = \mathrm{e}$ will not be used. The minimal number is the length $\ell(\sigma)$ of the permutation σ. Hence there is a well defined mapping $S_j \longrightarrow B_j$ given by $\sigma = t_{a_1}\, t_{a_2} \cdots t_{a_{\ell(\sigma)}} \mapsto \psi_{a_1}\, \psi_{a_2} \cdots \psi_{a_{\ell(\sigma)}}$, where the set $\{\psi_a\}_{a \in \{1,\ldots,j-1\}}$ consists of the elementary generators of the braid group B_j [Art]. The canonical composition $S_j \longrightarrow B_j \longrightarrow S_j$ is obviously the identity on S_j and therefore the

mapping $S_j \longrightarrow B_j$ is a section. Since the category $\mathcal{C}$ is braided, one can define the mapping $S_j \longrightarrow \mathrm{End}(X^{\otimes j})$ by $\sigma = t_{a_1}\, t_{a_2} \cdots t_{a_{\ell(\sigma)}} \mapsto \sigma_{\mathcal{C}}(X) := (\Psi_{X,X})_{a_1}\, (\Psi_{X,X})_{a_2} \cdots (\Psi_{X,X})_{a_{\ell(\sigma)}}$ where $(\Psi_{X,X})_a = \mathrm{id}_{X^{\otimes\, a-1}} \otimes \Psi_{X,X} \otimes \mathrm{id}_{X^{\otimes\, n-a-1}}$ for all $a \in \{1, \ldots, n-1\}$ [Wor]. Now let $\pi = (j_1, \ldots, j_r)$ be any $\mathbb{N}_0$-partition of j, i. e. $j = j_1 + \cdots + j_r$ and $j_1, \ldots j_r \in \mathbb{N}_0$. We consider the shuffle permutations $S^j_\pi \subset S_j$. For every $k \in \{1, \ldots, r\}$ they are mapping j_k elements of $\{1, \ldots, j\}$ to $\{(\sum_{l=1}^{k-1} j_l) + 1, \ldots, (\sum_{l=1}^{k} j_l)\}$ without changing their order. The set of inverse permutations of S^j_π is denoted by S^π_j. Based on the ideas of [Ma4, Wor] we define braided multinomials for every partition $\pi = (j_1, \ldots, j_r)$ of j and any object $X \in \mathrm{Ob}(\mathcal{C})$ according to

$$\begin{bmatrix} \pi \\ j \end{bmatrix}\!\Big| X; \lambda \Big] = \begin{bmatrix} j_1 \ldots j_r \\ j \end{bmatrix}\!\Big| X; \lambda \Big] := \sum_{\sigma \in S^\pi_j} \lambda^{\ell(\sigma)}\, \sigma_{\mathcal{C}}(X)\,. \tag{3.19}$$

where $\lambda \in \mathrm{Aut}\,(\mathbf{1})$ is an automorphism of the unit object in $\mathcal{C}$. The dual counterparts of (3.19) are given by

$$\begin{bmatrix} j \\ \pi \end{bmatrix}\!\Big| X; \lambda \Big] = \begin{bmatrix} j \\ j_1 \ldots j_r \end{bmatrix}\!\Big| X; \lambda \Big] := \sum_{\sigma \in S^j_\pi} \lambda^{\ell(\sigma)}\, \sigma_{\mathcal{C}}(X)\,. \tag{3.20}$$

Both types are endomorphisms of $X^{\otimes j}$ in $\mathcal{C}$. It is not difficult to prove that $\left[{j \atop j}\middle| X; \lambda\right] = \mathrm{id}_{X^{\otimes j}}$, and $\left[j\middle| X; \lambda\right]! := \left[{j \atop 1\ldots 1}\middle| X; \lambda\right] = \left[{1\ldots 1 \atop j}\middle| X; \lambda\right]$ which corresponds to the antisymmetrizer A_j in [Wor]. We obtain the braided form of the fundamental result on multinomials of subpartitions.

Proposition 3.8 *Let $\pi = (j_1, \ldots, j_r)$ be a partition of j, and let $\pi_k = (j^k_1, \ldots, j^k_{r_k})$ be a partition of j_k for any $k \in \{1, \ldots, r\}$. With the notation from above it holds*

$$\begin{bmatrix} (\pi_1, \pi_2, \ldots, \pi_r) \\ j \end{bmatrix}\!\Big| X; \lambda \Big] = \begin{bmatrix} \pi \\ j \end{bmatrix}\!\Big| X; \lambda \Big] \circ \left(\begin{bmatrix} \pi_1 \\ j_1 \end{bmatrix}\!\Big| X; \lambda \Big] \otimes \begin{bmatrix} \pi_2 \\ j_2 \end{bmatrix}\!\Big| X; \lambda \Big] \otimes \cdots \otimes \begin{bmatrix} \pi_r \\ j_r \end{bmatrix}\!\Big| X; \lambda \Big] \right) \tag{3.21}$$

and

$$\begin{bmatrix} j \\ (\pi_1, \pi_2, \ldots, \pi_r) \end{bmatrix}\!\Big| X; \lambda \Big] = \left(\begin{bmatrix} j_1 \\ \pi_1 \end{bmatrix}\!\Big| X; \lambda \Big] \otimes \begin{bmatrix} j_2 \\ \pi_2 \end{bmatrix}\!\Big| X; \lambda \Big] \otimes \cdots \otimes \begin{bmatrix} j_r \\ \pi_r \end{bmatrix}\!\Big| X; \lambda \Big] \right) \circ \begin{bmatrix} j \\ \pi \end{bmatrix}\!\Big| X; \lambda \Big]\,. \tag{3.22}$$

Equation (3.22) is the dual version of (3.21).

Proof. Let $j \in \mathbb{N}_0$ and $\pi = (j_1, \ldots, j_r)$ be a partition of j. Similarly as in [Wor] one proves that every permutation $\sigma \in S_j$ decomposes uniquely according to

$$\sigma = \sigma_{(j_1, \ldots, j_r)} \circ p_1 \circ \cdots \circ p_r \tag{3.23}$$

where $\sigma_{(j_1, \ldots, j_r)} \in S^j_\pi$ and for any $k \in \{1, \ldots, r\}$ the p_k permutes the elements $\{(\sum_{l=1}^{k-1} j_l) + 1, \ldots, (\sum_{l=1}^{k} j_l)\} \subset \{1, \ldots, j\}$ only, i. e. p_k is the tensor product (in the category of symmetric groups) of the identity in $S_{j_1 + \cdots + j_{k-1}}$ with a permutation in S_{j_k} and the identity in $S_{j-(j_1+\cdots+j_k)}$. Moreover the length of σ is given by $\ell(\sigma) = \ell(\sigma_{(j_1, \ldots, j_r)}) + \ell(p_1) + \cdots + \ell(p_r)$. Suppose that $\pi_k = (j^k_1, \ldots, j^k_{r_k})$ is a partition of j_k for any $k \in \{1, \ldots, r\}$. Then every $\sigma \in S^j_{(\pi_1, \ldots, \pi_r)}$ decomposes uniquely like in (3.23) where the nontrivial action of the permutation p_k is given by a shuffle permutation in $S^{j_k}_{\pi_k}$. Inserting these results in the Definition 3.19 for $\left[{j \atop (\pi_1, \ldots, \pi_r)}\middle| X; \lambda\right]$ yields (3.22). The dual statement (3.21) is obtained similarly. ■

For any $0 \leq k \leq j$ we define the morphisms $\left[{j \atop k}\middle|X;\lambda\right] := \left[{j \atop k,j-k}\middle|X;\lambda\right]$ and $\left[{k \atop j}\middle|X;\lambda\right] := \left[{k,j-k \atop j}\middle|X;\lambda\right]$. Then we derive by successive application of eq. (3.22) the identities (in abbreviated form)

$$
\begin{aligned}
[j]! &= \left(\mathrm{id}_{X^{\otimes\, j-2}} \otimes \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \circ \left(\mathrm{id}_{X^{\otimes\, j-3}} \otimes \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) \circ \cdots \circ \begin{bmatrix} j \\ 1 \end{bmatrix}, \\
[j]! &= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \otimes \mathrm{id}_{X^{\otimes\, j-2}}\right) \circ \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} \otimes \mathrm{id}_{X^{\otimes\, j-3}}\right) \circ \cdots \circ \begin{bmatrix} j \\ j-1 \end{bmatrix}
\end{aligned}
\tag{3.24}
$$

and

$$
[j+k]! = ([j]! \otimes [k]!) \circ \begin{bmatrix} j+k \\ j \end{bmatrix}. \tag{3.25}
$$

Similarly the equations which are dual to (3.24) and (3.25) can be deduced. For the first equation in (3.24) see also [Ma4]. We have provided all the definitions and results which are necessary for the formulation and the proof of the proposition on tensor Hopf algebras in braided categories extending the corresponding theorem in [Ma5].

Proposition 3.9 *The tensor algebra $T_{\mathcal{C}}(X)$ in $\mathcal{C}^{\mathbb{N}}$ of an object X in a $\otimes$-factor braided abelian category $\mathcal{C}$ has a Hopf algebra structure which is given by*

$$
\begin{aligned}
\mathrm{m}_{n,m} &\cong \mathrm{id}_{X^{\otimes\, n+m}} : X^{\otimes\, n} \otimes X^{\otimes\, m} \longrightarrow X^{\otimes\, n+m}, \\
\eta_0 &= \mathrm{id}_k,\ \eta_j = 0 \text{ for } j \neq 0, \\
\Delta_{n,m} &\cong \left[{n+m \atop n}\middle|X;\lambda\right] : X^{\otimes\, n+m} \longrightarrow X^{\otimes\, n} \otimes X^{\otimes\, m}, \\
\varepsilon_0 &= \mathrm{id}_k,\ \varepsilon_j = 0 \text{ for } j \neq 0, \\
S_n &= (-1)^n \lambda^{\binom{n}{2}} (\sigma_n^0)_{\mathcal{C}}(X) : X^{\otimes\, n} \longrightarrow X^{\otimes\, n}
\end{aligned}
\tag{3.26}
$$

where $\sigma_n^0 = \left({1 \atop n}{\cdots \atop \cdots}{n \atop 1}\right)$. Because of duality reasons another Hopf structure can be established on $T_{\mathcal{C}}(X)$ according to

$$
\begin{aligned}
\mathring{\mathrm{m}}_{n,m} &\cong \left[{n \atop n+m}\middle|X;\lambda\right] : X^{\otimes\, n} \otimes X^{\otimes\, m} \longrightarrow X^{\otimes\, n+m}, \\
\mathring{\eta}_0 &= \mathrm{id}_k,\ \mathring{\eta}_j = 0 \text{ for } j \neq 0, \\
\mathring{\Delta}_{n,m} &\cong \mathrm{id}_{X^{\otimes\, n+m}} : X^{\otimes\, n+m} \longrightarrow X^{\otimes\, n} \otimes X^{\otimes\, m}, \\
\mathring{\varepsilon}_0 &= \mathrm{id}_k,\ \mathring{\varepsilon}_j = 0 \text{ for } j \neq 0, \\
\mathring{S}_n &= (-1)^n \lambda^{\binom{n}{2}} (\sigma_n^0)_{\mathcal{C}}(X) : X^{\otimes\, n} \longrightarrow X^{\otimes\, n}.
\end{aligned}
\tag{3.27}
$$

For distinction we denote by $T_{\mathcal{C}}(X)$ the Hopf algebra defined through (3.26). The dual Hopf algebra corresponding to (3.27) will be denoted by $\mathring{T}_{\mathcal{C}}(X)$.

Proof. Using the results on braided multinomials the proof of the proposition is not difficult. To give an insight into the techniques we sketch the proof of the Hopf algebra structure (3.26). Without problems one verifies the algebra structure of (3.26). For the proof of the coalgebra properties one observes that Proposition 3.8 implies

$$
\left(\begin{bmatrix} m+n \\ m \end{bmatrix} \otimes \mathrm{id}_{X^{\otimes\, k}}\right) \circ \begin{bmatrix} m+n+k \\ m+n \end{bmatrix} = \left(\mathrm{id}_{X^{\otimes\, m}} \otimes \begin{bmatrix} n+k \\ n \end{bmatrix}\right) \circ \begin{bmatrix} m+n+k \\ m \end{bmatrix} \tag{3.28}
$$

which yields the coassociativity. A similar investigation as in the proof of Proposition 3.8 leads to the relation

$$
\begin{bmatrix} n+m+p+q \\ n+p \end{bmatrix} = (\mathrm{id}_{X^{\otimes\, n}} \otimes \Psi^{(\lambda)}_{X^{\otimes\, m}, X^{\otimes\, p}} \otimes \mathrm{id}_{X^{\otimes\, q}}) \circ \left(\begin{bmatrix} n+m \\ n \end{bmatrix} \otimes \begin{bmatrix} p+q \\ p \end{bmatrix}\right). \tag{3.29}
$$

Then it can be shown directly that $\hat{\Delta}$ in (3.26) is an algebra morphism, that is

$$\hat{\Delta} \circ \hat{\mathrm{m}} = (\hat{\mathrm{m}} \otimes \hat{\mathrm{m}}) \circ (\hat{\mathrm{id}} \otimes \hat{\Psi}^{(\lambda)}_{T,T} \otimes \hat{\mathrm{id}}) \circ (\hat{\Delta} \otimes \hat{\Delta}) . \tag{3.30}$$

Using (3.29) and the relation $S_n = -\Psi^{(\lambda)}_{X,X^{\otimes,n-1}} \circ (\mathrm{id}_X \otimes S_{n-1})$ we verify the Hopf equation $\hat{\mathrm{m}} \circ (\hat{\mathrm{id}} \otimes \hat{S}) \circ \hat{\Delta} = \hat{\eta} \circ \hat{\varepsilon}$ by induction on n, and analogously we derive $\hat{\mathrm{m}} \circ (\hat{S} \otimes \hat{\mathrm{id}}) \circ \hat{\Delta} = \hat{\eta} \circ \hat{\varepsilon}$. ■

Up to now braided multinomials are involved which generally depend on an automorphism λ of the unit object in $\mathcal{C}$. However, the study of the differential structure in Chapter 4 forces us to consider $\lambda = -1$. Therefore we focus our consideration to $\lambda = -1$ without serious loss of generality. In what follows we will use the notation $\left[\begin{smallmatrix} j_1 \ldots j_r \\ j \end{smallmatrix} \middle| X\right] := \left[\begin{smallmatrix} j_1 \ldots j_r \\ j \end{smallmatrix} \middle| X; (-1)\right]$ and $\left[\begin{smallmatrix} j \\ j_1 \ldots j_r \end{smallmatrix} \middle| X\right] := \left[\begin{smallmatrix} j \\ j_1 \ldots j_r \end{smallmatrix} \middle| X; (-1)\right]$ for the braided multinomials, and $\hat{\Psi} := \hat{\Psi}^{(-1)}$ for the braiding in $\mathcal{C}^{\underline{\mathbb{N}}}$.

Definition 3.10 *Let X be an arbitrary object in the category $\mathcal{C}$. Then the antisymmetrizer $\hat{A}_X : T_{\mathcal{C}}(X) \longrightarrow T_{\mathcal{C}}(X)$ is given by $(\hat{A}_X)_n = [n|X]! : X^{\otimes\, n} \longrightarrow X^{\otimes\, n}$.*

The antisymmetrizer $\hat{A}_X$ is compatible with the Hopf structures of $T_{\mathcal{C}}(X)$ and $\mathring{T}_{\mathcal{C}}(X)$.

Proposition 3.11 *Let X be an object in $\mathcal{C}$. Then $\hat{A}_X : T_{\mathcal{C}}(X) \longrightarrow \mathring{T}_{\mathcal{C}}(X)$ is a Hopf algebra morphism.*

Proof. The proof is straightforward. We will only outline that $\hat{A}_X$ is an algebra morphism. We use the relation dual to (3.25).

$$\mathring{\mathrm{m}}_{n,m} \circ (A_{X,n} \otimes A_{X,m}) = \begin{bmatrix} n \\ n+m \end{bmatrix} \circ \big([n]! \otimes [m]!\big) = [n+m]! \cong A_{X,n+m} \circ \mathrm{m}_{n,m} \, . \tag{3.31}$$

■

Proposition 3.11 and Lemma 2.4 lead to the definiton of the braided antisymmetric tensor algebra.

Definition 3.12 *Let X be an object in the category $\mathcal{C}$. Then the antisymmetric tensor algebra of X is the Hopf algebra $T^{\wedge}_{\mathcal{C}}(X)$ in $\mathcal{C}^{\underline{\mathbb{N}}}$ canonically induced by the decomposition $\hat{A}_X = \mathrm{im}\, \hat{A}_x \circ \mathrm{coim}\, \hat{A}_x$.*

$$T_{\mathcal{C}}(X) \xrightarrow{\mathrm{coim}\, \hat{A}_X} T^{\wedge}_{\mathcal{C}}(X) \xrightarrow{\mathrm{im}\, \hat{A}_X} \mathring{T}_{\mathcal{C}}(X) \, . \tag{3.32}$$

Definition 3.12 suggests to define the functors $T_{\mathcal{C}}(\_)$, $\mathring{T}_{\mathcal{C}}(\_)$, $T^{\wedge}_{\mathcal{C}}(\_) : \mathcal{C} \longrightarrow$ Hopf-alg-$\mathcal{C}^{\underline{\mathbb{N}}}$ on the objects according to (3.32). On morphisms $T_{\mathcal{C}}(\_)$ and $\mathring{T}_{\mathcal{C}}(\_)$ are given canonically, and for any morphism $g : X \longrightarrow Y$ in $\mathcal{C}$ the morphism $T^{\wedge}_{\mathcal{C}}(g) : T^{\wedge}_{\mathcal{C}}(X) \longrightarrow T^{\wedge}_{\mathcal{C}}(Y)$ is defined to be the unique morphism such that $\mathrm{im}\, \hat{A}_Y \circ T^{\wedge}_{\mathcal{C}}(g) = \mathring{T}_{\mathcal{C}}(g) \circ \mathrm{im}\, \hat{A}_X$. Then the following result can be derived which will be needed in Chapter 4 for the construction of a graded differential Hopf algebra out of a first order bicovariant differential calculus.

Lemma 3.13 *Let $\mathcal{C}$ be a $\otimes$-factor braided abelian category. Then the functors $T_{\mathcal{C}}(\_)$, $T^{\wedge}_{\mathcal{C}}(\_)$ and $\overset{\circ}{T}_{\mathcal{C}}(\_)$ map epi-/monomorphisms to epi-/monomorphisms. Every morphism $g : X \longrightarrow Y$ in $\mathcal{C}$ yields $\big(T_{\mathcal{C}}(g), T^{\wedge}_{\mathcal{C}}(g), \overset{\circ}{T}_{\mathcal{C}}(g)\big)$ which is a morphism of sequences of the form (3.32).*

Proof. Obviously for any $g : X \longrightarrow Y$ in $\mathcal{C}$ the triple $\big(T_{\mathcal{C}}(g), T^{\wedge}_{\mathcal{C}}(g), \overset{\circ}{T}_{\mathcal{C}}(g)\big)$ is a morphism of the corresponding diagrams (3.32). The assertion on the epi-/monomorphism property is easily verified for $T_{\mathcal{C}}(\_)$ and $\overset{\circ}{T}_{\mathcal{C}}(\_)$. Then the commutativity of the diagrams involving the morphisms $\big(T_{\mathcal{C}}(g), T^{\wedge}_{\mathcal{C}}(g), \overset{\circ}{T}_{\mathcal{C}}(g)\big)$ yields the statement also for $T^{\wedge}_{\mathcal{C}}(\_)$. ∎

For a flat Hopf algebra H with isomorphic antipode in the $\otimes$-factor braided abelian category $\mathcal{C}$ we consider the category $({}^H_H\mathcal{C}^H_H)^{\underline{\mathbb{N}}}$ which is again $\otimes$-factor braided abelian due to Proposition A.4. In analogy to Definition 3.12 the functors $T_{{}^H_H\mathcal{C}^H_H}(\_)$, $\overset{\circ}{T}_{{}^H_H\mathcal{C}^H_H}(\_)$, $T^{\wedge}_{{}^H_H\mathcal{C}^H_H}(\_) : {}^H_H\mathcal{C}^H_H \longrightarrow ({}^H_H\mathcal{C}^H_H)^{\underline{\mathbb{N}}}$ can be defined. Then we apply Theorem 1.5 to produce Hopf algebras in $\mathcal{C}^{\underline{\mathbb{N}}}$.

Definition 3.14 *Given a flat Hopf algebra H with bijective antipode in $\mathcal{C}$ and an H-Hopf bimodule X. Then the tensor Hopf algebras $\mathcal{T}^H_{\mathcal{C}}(X)$, $\overset{\circ}{\mathcal{T}}{}^H_{\mathcal{C}}(X)$ and the braided exterior tensor Hopf algebra of forms $X^{\wedge_H}$ over X in $\mathcal{C}^{\underline{\mathbb{N}}}$ are defined as $\mathcal{T}^H_{\mathcal{C}}(X) := G(T_{{}^H_H\mathcal{C}^H_H}(X))$, $\overset{\circ}{\mathcal{T}}{}^H_{\mathcal{C}}(X) := G(\overset{\circ}{T}_{{}^H_H\mathcal{C}^H_H}(X))$ and $X^{\wedge_H} := G(T^{\wedge}_{{}^H_H\mathcal{C}^H_H}(X))$ respectively, where the corresponding functor G of Theorem 1.5 is used.*

Remark 13 The Theorems 1.4 and 1.5 and the results of the appendix may be used to prove that

$$X^{\wedge_H} \cong G\big(H \ltimes T^{\wedge}_{\mathcal{DY}(\mathcal{C})^H_H}({}_H X)\big) \tag{3.33}$$

are isomorphic Hopf algebra objects. In anticipation of Chapter 4 suppose that (X, d) is a braided first order bicovariant differential calculus over H. Then the object $(X^{\wedge_H}, \mathrm{d}^{\wedge})$ is the braided exterior Hopf algebra of differential forms over X, and (3.33) extends to an isomorphism of differential Hopf algebras

$$(X^{\wedge_H}, \mathrm{d}^{\wedge}) \cong \big(G\big(H \ltimes T^{\wedge}_{\mathcal{DY}(\mathcal{C})^H_H}({}_H X)\big), H \ltimes {}_H(\hat{\mathrm{d}})\big)\,. \tag{3.34}$$

This particularly implies the results (in the classical symmetric situation) of [SZ] and dually of [Dra]. There it was found that the higher order differential calculi of [Wor] and the quantum standard complexes are cross products of the quantum group and the quantum enveloping algebra respectively with a certain antisymmetric tensor algebra of invariant vector fields.

Under the conditions of Definition 3.14 we know from Proposition 3.6 that (H, X) is a bialgebra in $\mathcal{C}^{\mathbf{2}}$. By construction of the tensor algebras $\mathcal{T}^H_{\mathcal{C}}(X)$ and $\overset{\circ}{\mathcal{T}}{}^H_{\mathcal{C}}(X)$ we derive from Theorem 1.5 and the corresponding definition in (1.13) that the multiplication of $\mathcal{T}^H_{\mathcal{C}}(X)$ is given by $\mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)} = \lambda^H_{\mathcal{T}^H_{\mathcal{C}}(X), \mathcal{T}^H_{\mathcal{C}}(X)}$ and the comultiplication of $\overset{\circ}{\mathcal{T}}{}^H_{\mathcal{C}}(X)$ equals $\Delta^{\overset{\circ}{\mathcal{T}}{}^H_{\mathcal{C}}(X)} = \rho^H_{\overset{\circ}{\mathcal{T}}{}^H_{\mathcal{C}}(X), \overset{\circ}{\mathcal{T}}{}^H_{\mathcal{C}}(X)}$. This already indicates the universality of the tensor algebras which will be demonstrated in the following theorems and in Theorem 4.21.

Proposition 3.15 *Suppose that H is a flat Hopf algebra with isomorphic antipode in $\mathcal{C}$ and X is an H-Hopf bimodule.*

If $\hat{Y}$ be an algebra resp. a bi-/Hopf algebra in $\mathcal{C}^{\underline{\mathbb{N}}}$ and $\hat{f} = (f_0, f_1) : (A, X) \longrightarrow (Y_0, Y_1)$ is an algebra resp. bialgebra morphism in $\mathcal{C}^{\{0,1\}}$ then there exitsts a unique algebra resp. bialgebra morphism $\hat{\hat{f}} : \mathcal{T}^A_{\mathcal{C}}(X) \longrightarrow \hat{Y}$ such that $(\hat{\hat{f}})_0 = f_0$ and $(\hat{\hat{f}})_1 = f_1$.

A dually symmetric result holds for a coalgebra resp. bialgebra morphism $\mathring{g} = (g_0, g_1) : (Y_0, Y_1) \longrightarrow (H, X)$ which can be extended uniquely to a coalgebra resp. bialgebra morphism $\hat{\mathring{g}} : \hat{Y} \longrightarrow \mathring{\mathcal{T}}^H_{\mathcal{C}}(X)$ with $(\hat{\mathring{g}})_0 = g_0$ and $(\hat{\mathring{g}})_1 = g_1$.

Proof. Since $(f_0, f_1) : (H, X) \longrightarrow (Y_0, Y_1)$ is an algebra morphism by assumption and the multiplication of $\mathcal{T}^H_{\mathcal{C}}(X)$ is given by the universal tensor product morphism $\mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)} = \lambda^H_{\mathcal{T}^H_{\mathcal{C}}(X), \mathcal{T}^H_{\mathcal{C}}(X)}$ one immediately verifies successively that there exists a unique morphism $f_n : X^{\otimes_H n} \longrightarrow Y_n$ for every $n \in \mathbb{N}_0$ such that

$$\mathrm{m}^Y_{\underbrace{1, \ldots , 1}_{n}} \circ (\underbrace{f_1 \otimes \ldots \otimes f_1}_{n}) = f_n \circ \mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{\underbrace{1, \ldots , 1}_{n}} .$$

Then the identity $f_{m+n} \circ \mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{m,n} \circ (\mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{1,\ldots,1} \otimes \mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{1,\ldots,1}) = \mathrm{m}^Y_{m,n} \circ (f_m \otimes f_n) \circ (\mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{1,\ldots,1} \otimes \mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{1,\ldots,1})$ can be derived which implies $f_{m+n} \circ \mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{m,n} = \mathrm{m}^Y_{m,n} \circ (f_m \otimes f_n)$ because the morphism $\mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{1,\ldots,1}$ is epimorphic. Thus $\hat{\hat{f}} := (f_n) : \mathcal{T}^H_{\mathcal{C}}(X) \longrightarrow \hat{Y}$ is an algebra morphism.

If $\hat{Y}$ is a bi-/Hopf algebra and $(f_0, f_1) : (H, X) \longrightarrow (Y_0, Y_1)$ is a bialgebra morphism, it follows by induction on $n > 1$ and by use of the fact that $\hat{\hat{f}}$ is an algebra morphism, that

$$\Delta^{\hat{Y}}_{t,n-t} \circ f_n \circ \mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{n-1,1} = (f_t \otimes f_{n-t}) \circ \Delta^{\mathcal{T}^H_{\mathcal{C}}(X)}_{t,n-t} \circ \mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{n-1,1} .$$

Hence $\hat{\hat{f}}$ is a bialgebra morphism since $\mathrm{m}^{\mathcal{T}^H_{\mathcal{C}}(X)}_{n-1,1}$ is epimorphic. ■

Remark 14 Note that under the assumption of right exactness of $X \otimes (\_)$ for every $X \in \mathrm{Ob}(\mathcal{C})$ for any algebra A in $\mathcal{C}$ the category ${}_A\mathcal{C}_A$ of A-bimodules with the tensor product $\otimes_A$ over A is monoidal. And one can define a free tensor algebra $\mathcal{T}^A_{\mathcal{C}}(X)$ for every A-bimodule X with multiplication given by the canonical morphisms into the tensor product over A. This tensor algebra is characterized by the same universal property as described in Proposition 3.15.

Consider as a special example of Proposition 3.15 the Hopf algebra $\hat{Y} := \mathring{\mathcal{T}}^H_{\mathcal{C}}(X)$. Since $(\mathrm{id}, \mathrm{id}) : (H, X) \longrightarrow (H, X)$ is a bialgebra morphism we obtain with the help of Proposition 3.15 the following corollary.

Corollary 3.16 *The unique higher order bialgebra extension of the identity morphism $\hat{\mathrm{id}} : (H, X) \longrightarrow (H, X)$ is given by $\hat{\hat{\mathrm{id}}} = \hat{A}^H_X : \mathcal{T}^H_{\mathcal{C}}(X) \longrightarrow \mathring{\mathcal{T}}^H_{\mathcal{C}}(X)$ where $\hat{A}^H_X$ is the antisymmetrizer of X in the category $({}^H_H\mathcal{C}^H_H)^{\underline{\mathbb{N}}}$.* ■

The main theorem in Chapter 3 states the universal property of the exterior Hopf algebra $X^{\wedge_H}$ generated algebraically by its 0th and 1st component.

Theorem 3.17 *Suppose that H is a flat Hopf algebra with isomorphic antipode and X is an H-Hopf bimodule in $\mathcal{C}$. Then the exterior Hopf algebra $X^{\wedge_H}$ is generated by H and X as an algebra, that is $\mathrm{Im}(\mathrm{m}_{1,n}^{X^{\wedge_H}}) = X^{\wedge_H}_{n+1}$ for any $n \in \mathbb{N}_0$. Let $\hat{Y}$ be a bi-/Hopf algebra in $\mathcal{C}^{\underline{\mathbb{N}}}$. Suppose that $\hat{Y}$ is generated by Y_0 and Y_1 as an algebra. If $(g_0, g_1) : (Y_0, Y_1) \longrightarrow (H, X)$ is a bialgebra morphism in $\mathcal{C}^{\mathbf{2}}$, then there exists a unique bialgebra morphism $g^{\wedge_H} : \hat{Y} \longrightarrow X^{\wedge_H}$ in $\mathcal{C}^{\underline{\mathbb{N}}}$ such that $(g^{\wedge_H})_0 = g_0$ and $(g^{\wedge_H})_1 = g_1$. In this case it holds $\hat{\mathring{g}} = \mathrm{im}\, \hat{A}^H_X \circ g^{\wedge_H}$.*

Proof. On $X^{\wedge_H}$ the multiplication is given by $\mathrm{m}^{X^{\wedge_H}} = \mathrm{m}_{\mathrm{Coim}\, \hat{A}} \circ \lambda^H_{\mathrm{Coim}\, \hat{A}, \mathrm{Coim}\, \hat{A}}$ according to the construction in Theorem 1.5. Then it follows

$$\mathrm{m}_{X^{\wedge_H}} \circ (\mathrm{coim}\, \hat{A} \otimes \mathrm{coim}\, \hat{A}) = \mathrm{coim}\, \hat{A} \circ \lambda^H_{T_{{}^H_H\mathcal{C}^H_H}(X), T_{{}^H_H\mathcal{C}^H_H}(X)} \tag{3.35}$$

and since the right hand side of (3.35) is epimorphic, one deduces that $X^{\wedge_H}$ is generated algebraically by H and X. Now let us suppose that there exists a bialgebra morphism $(g_0, g_1) : (Y_0, Y_1) \longrightarrow (H, X)$. Proposition 3.15 and Corollary 3.16 then tell us that there is a bialgebra morphism $\hat{\mathring{g}} : \hat{Y} \longrightarrow \mathring{\mathcal{T}}^H_{\mathcal{C}}(X)$. It holds $\mathrm{coker}(\hat{A}) \circ \hat{\mathring{g}} = 0$ which can be seen by induction as follows. For $n = 0, 1$ one verifies immediately that $(\mathrm{coker}\, \hat{A})_n \circ (\hat{\mathring{g}})_n = 0$ since $(\hat{A})_0 = \mathrm{id}$ and $(\hat{A})_1 = \mathrm{id}$. Therefore $g_0 = (\hat{\mathring{g}})_0 = (\mathrm{im}\, \hat{A})_0 \circ g_0^{\wedge} =: g_0^{\wedge}$ and $g_1 = (\hat{\mathring{g}})_1 = (\mathrm{im}\, \hat{A})_1 \circ g_1^{\wedge} =: g_1^{\wedge}$. If $n > 1$ one derives the relation $(\mathrm{coker}\, \hat{A})_n \circ (\hat{\mathring{g}})_n \circ \mathrm{m}^{\hat{Y}}_{n-1,1} = 0$ because $(\hat{\mathring{g}})$ is a bialgebra morphism, $(\hat{\mathring{g}})_m = (\mathrm{im}\, \hat{A})_m \circ g_m^{\wedge}$ for every $m < n$, and $(\mathrm{im}\, \hat{A})$ is a bialgebra morphism. Since $\mathrm{m}^{\hat{Y}}_{n-1,1}$ is epimorphic by assumption, we obtain $(\hat{\mathring{g}})_n = (\mathrm{im}\, \hat{A})_n \circ g_n^{\wedge}$ for some unique $g_n^{\wedge}$. ■

Remark 15 In the case of the antisymmetric and exterior Hopf algebras it is not sufficient in general to restrict one's considerations to the grades less than 3 and to argue by multiplicative continuation as will be demonstrated in the following. Suppose that the category $\mathcal{C}$ is $\otimes$-exact braided abelian and $X \otimes \mathrm{id}_{\mathcal{C}}$ is right exact. Consider the ideal $\big(\ker[2|X;\lambda]!\big)$ in $T_{{}^H_H\mathcal{C}^H_H}(X)$ generated by $\ker[2|X;\lambda]!$ according to Proposition 2.9. Again λ is an automorphism of the unit object. It is not difficult to verify that $\big(\ker[2|X;\lambda]!\big)$ is a Hopf ideal and a subobject of $\ker \hat{A}_X$ because $[n+m|X;\lambda]! = \big([n-2|X;\lambda]! \otimes [2|X;\lambda]! \otimes [m|X;\lambda]!\big) \circ \big[{n+m \atop (n-2,2,m)}|X;\lambda\big]$, and dually by Proposition 3.8 and eq. (3.25). But in general the ideal $\ker \hat{A}_X$ does not coincide with $\big(\ker[2|X;\lambda]!\big)$. For example the tensor algebra of polynomials in one variable $\mathbb{C}\langle x\rangle$ can be equipped with the braiding $\Psi(x \otimes x) = q(x \otimes x)$, $q \in \mathbb{C}^*$. This is the braided line as described in [Koo, Ma4]. The braided integers $[n|X;q]$ are just the q-numbers $[n|X;q] = [n]_q := 1 + q + \cdots + q^{n-1}$ and the braided factorial $[n|X;q]!$ equals $[n]_q!$ which is the usual q-factorial. For q a primitive root of unity of order $n > 2$ one obtains $(\ker[2|X;q]!) = 0$ which does not coincide with the ideal $\ker \hat{A}_X = (x^{\otimes n})$.

4 Differential Calculi and Exterior Differential Hopf Algebras

In Chapter 4 we define differential calculi in $\otimes$-factor abelian categories following the ideas of [Mal, Man, Wor]. Special interest is given to the generalization of bicovariant differential calculi in $\otimes$-factor braided abelian categories. We prove the existence of a higher order differential Hopf algebra calculus which extends a given first order bicovariant differential calculus in a natural way. This is the braided exterior Hopf algebra of differential forms

over the first order bicovariant differential calculus. The corresponding results of [Wor] are therefore generalized to $\otimes$-factor braided abelian categories.

Differential Calculi

Definition 4.1 *A complex $(\hat{X}, \hat{\mathrm{d}})$ in $\mathcal{C}^{\mathrm{I}_c}$ is called a differential calculus if $(\hat{X}, \hat{\mathrm{d}})$ is an algebra in $\mathcal{C}^{\mathrm{I}_c}$ and $X_0\langle \mathrm{d}_n\rangle = X_{n+1}$ for all $n \in \mathrm{I}$. The category of differential calculi as a full subcategory of $\mathrm{Alg}\text{-}\mathcal{C}^{\mathrm{I}_c}$ will be denoted by $\mathrm{Diff}\text{-}\mathcal{C}^{\mathrm{I}_c}$.*

Sometimes we adopt the notation of [Wor] and we speak of first order differential calculi if $\mathrm{I}_c = \mathbf{2}_c$ and of higher order differential calculi if $\mathrm{I}_c = \underline{\mathbb{N}}_c$. From Definition 4.1 we derive some implications in analogy to the results of [Wor] which we will collect in the next propositions.

Proposition 4.2 *The following statements for an algebra $(\hat{X}, \hat{\mathrm{d}})$ in $\mathcal{C}^{\mathrm{I}_c}$ are equivalent.*

1. *$(\hat{X}, \hat{\mathrm{d}})$ is a differential calculus.*
2. *For all $n \in \mathrm{I}$ it holds $X_0\langle \mathrm{d}_n\rangle = X_{n+1}$.*
3. *For all $n \in \mathrm{I}$ it holds $\langle \mathrm{d}_n\rangle X_0 = X_{n+1}$.*
4. *For all $n \in \mathrm{I}$ it holds $X_0\langle \mathrm{d}_n\rangle X_0 = X_{n+1}$.*
5. *For all $n \in \mathrm{I}$ it holds $X_0\langle \mathrm{m}^{(n-1)}_{1,\dots,1} \circ (\mathrm{d} \otimes \ldots \otimes \mathrm{d})\rangle = X_{n+1}$.*

The differential of a differential calculus $(\hat{X}, \hat{\mathrm{d}})$ is determined uniquely by its 0th component d_0.

Proof. The proof of the equivalences is rather straightforward. It can be performed as in the symmetric category of vector spaces [Wor] because no braidings are involved. To give an idea of the techniques used in the proof we will demonstrate the equivalence of (2) and (3). We only have to prove that (2) implies (3). Because of symmetry reasons one can argue similarly that (2) follows from (3).

Suppose that $\mathrm{m}_{0,l} \circ (\mathrm{id}_0 \otimes \mathrm{d}_{l-1})$ is epimorphic. Then it follows by induction, using the identity

$$\begin{aligned} &\mathrm{m}_{k,l+m+1} \circ \big(\mathrm{id}_k \otimes \mathrm{d}_{l+m} \circ \mathrm{m}_{m,l} \circ (\mathrm{id}_m \otimes \mathrm{d}_{l-1})\big) \\ &= \mathrm{m}_{k+m+1,l} \circ (\mathrm{id}_{k+m+1} \circ \mathrm{d}_{l-1}) \circ \big(\mathrm{m}_{k,m+1} \circ (\mathrm{id}_k \otimes \mathrm{d}_m) \otimes \mathrm{id}_{l-1}\big) \end{aligned} \tag{4.1}$$

that $\mathrm{m}_{k,l} \circ (\mathrm{id}_k \otimes \mathrm{d}_{l-1})$ is epimorphic. $(\hat{X}, \hat{\mathrm{d}})$ is an algebra in $\mathcal{C}^{\mathrm{I}_c}$ which implies the relation

$$\mathrm{d}_n \circ \mathrm{m}_{k,l} = \mathrm{m}_{k+1,l} \circ (\mathrm{d}_k \otimes \mathrm{id}_l) + (-1)^k \mathrm{m}_{k,l+1} \circ (\mathrm{id}_k \otimes \mathrm{d}_l)\,. \tag{4.2}$$

Suppose that $f \circ \mathrm{m}_{k+1,0} \circ (\mathrm{d}_k \otimes \mathrm{id}_0) = 0$ for some morphism f. Then $f \circ \mathrm{d}_n \circ \mathrm{m}_{k,0} \circ (\mathrm{id}_k \otimes \eta_0) = (-1)^k f_{k+1} \circ (\mathrm{id}_k \otimes \mathrm{d}_0 \circ \eta_0) = 0$ because (4.2) and $\mathrm{d}_0 \circ \eta_0 = 0$ hold. Therefore $f \circ \mathrm{m}_{k,1} \circ (\mathrm{id}_k \otimes \mathrm{d}_0) = 0$ which yields $f = 0$. Hence $\mathrm{m}_{k+1,0} \circ (\mathrm{d}_k \otimes \mathrm{id}_0)$ is epimorphic. ■

Proposition 4.3 *Let $(\hat{X}, \hat{\mathrm{d}}_X)$ be a differential calculus in $\mathcal{C}^{\mathrm{I}_c}$ and $(\hat{Y}, \hat{\mathrm{d}}_Y)$ be an algebra in $\mathcal{C}^{\mathrm{I}_c}$. Then any algebra morphism $\hat{f} : (\hat{X}, \hat{\mathrm{d}}_X) \longrightarrow (\hat{Y}, \hat{\mathrm{d}}_Y)$ is uniquely determined by its 0th component f_0.*

Proof. From $f_n \circ \mathrm{m}^X_{0,n} \circ (\mathrm{id}_0 \otimes \mathrm{d}_{X,n-1}) = \mathrm{m}^Y_{0,n} \circ (\mathrm{id}_0 \otimes \mathrm{d}_{Y,n-1}) \circ (f_0 \otimes f_{n-1})$ the proof can be concluded since $\mathrm{m}^X_{0,n} \circ (\mathrm{id}_0 \otimes \mathrm{d}_{X,n-1})$ is epimorphic. ■

If we impose further conditions on $\mathcal{C}$ we can prove the converse as will be seen in the following proposition.

Proposition 4.4 *Suppose that $\mathcal{C}$ is $\otimes$-exact braided abelian. Let $(\hat{X}, \hat{\mathrm{d}}_X)$ be an algebra in $\mathcal{C}^{\mathrm{I_c}}$, and let the functor $\hat{X} \otimes \mathrm{id}_{\mathcal{C}^{\mathrm{I}}}$ be right exact. Suppose that any algebra morphism $\hat{f} : (\hat{X}, \hat{\mathrm{d}}_X) \longrightarrow (\hat{Y}, \hat{\mathrm{d}}_Y)$ in $\mathcal{C}^{\mathrm{I_c}}$ is uniquely determined by its 0th component f_0. Then $(\hat{X}, \hat{\mathrm{d}}_X)$ is a differential calculus in $\mathcal{C}^{\mathrm{I_c}}$.*

Proof. Define the object $\hat{Y}$ in $\mathcal{C}^{\mathrm{I}}$ by $Y_{n+1} = (\hat{X} \otimes \hat{X} \otimes \hat{X})_n$ for all $n \in \mathrm{I}$. For $m < 0$ we set $X_m = 0$ and $\mathrm{d}_{X,m} = 0$ in the following. We consider the morphism $\hat{M}_{\mathrm{d}} := \hat{\mathrm{m}}_{\hat{X}} \circ (\hat{\mathrm{id}}_{\hat{X}} \otimes \hat{\mathrm{m}}_{\hat{X}}) \circ (\hat{\mathrm{id}}_{\hat{X}} \otimes \hat{\mathrm{d}}_X \otimes \hat{\mathrm{id}}_{\hat{X}}) : \hat{Y} \longrightarrow \hat{X}$. It is the morphism which multiplies $\hat{X}$ from the left and from the right to $\hat{\mathrm{d}}_X(\hat{X})$. It holds $M_{\mathrm{d},0} = 0$. Because of Proposition 2.9 (in $\mathcal{C}^{\mathrm{I}}$) we know that $\hat{X}\langle\hat{\mathrm{d}}_X\rangle\hat{X} = \mathrm{im}\,\hat{M}_{\mathrm{d}}$ is an ideal in $\hat{X}$. With the help of Proposition 2.8 (in $\mathcal{C}^{\mathrm{I}}$) one observes that the morphism $\mathrm{coker}\,\hat{M}_{\mathrm{d}} : \hat{X} \longrightarrow \hat{P}$ is an algebra morphism. In particular $X_0 \cong P_0$. For $n > 0$ the equation $\mathrm{coker}(\hat{M}_{\mathrm{d}})_{n+1} \circ \mathrm{d}_{X,n} \circ \mathrm{im}(\hat{M}_{\mathrm{d}})_n = 0$ can be derived by use of the relation

$$\begin{aligned} &\mathrm{d}_{X,n} \circ \mathrm{m}^X_{k,l+m+1} \circ \big(\mathrm{id}_{X_k} \otimes \mathrm{m}^X_{l+1,m} \circ (\mathrm{d}_{X,l} \otimes \mathrm{id}_{X_m})\big) \\ &= \big((\hat{M}_{\mathrm{d}})_{k+1,l,m} \circ (\mathrm{d}_{X,k} \otimes \mathrm{id}_{X_l \otimes X_m}) + (-1)^{k+l+1} (\hat{M}_{\mathrm{d}})_{k,l,m+1} \circ (\mathrm{id}_{X_k \otimes X_l} \otimes \mathrm{d}_{X,m})\big) \end{aligned} \tag{4.3}$$

where $k + l + m = n - 1$. For $n = 0$ it holds $\mathrm{coker}(\hat{M}_{\mathrm{d}})_1 \circ \mathrm{d}_{X,0} = \mathrm{coker}(\hat{M}_{\mathrm{d}})_1 \circ (\hat{M}_{\mathrm{d}})_1 \circ (\eta_X \otimes \mathrm{id}_{X_0} \otimes \eta_X) = 0$. Hence there are unique morphisms $\mathrm{d}_{P,n} : P_n \longrightarrow P_{n+1}$ such that

$$\mathrm{d}_{P,n} \circ \mathrm{coker}(\hat{M}_{\mathrm{d}})_n = \mathrm{coker}(\hat{M}_{\mathrm{d}})_{n+1} \circ \mathrm{d}_{X,n}\,, \tag{4.4}$$

moreover $\mathrm{d}_{P,0} = 0$. From (4.4) it follows that $\mathrm{coker}(\hat{M}_{\mathrm{d}}) : (\hat{X}, \hat{\mathrm{d}}_X) \longrightarrow (\hat{P}, \hat{\mathrm{d}}_P)$ is an algebra morphism in $\mathcal{C}^{\mathrm{I_c}}$. Since $\mathrm{coker}(M_{\mathrm{d}})_0 = \mathrm{id}$ we conclude that $\mathrm{coker}(\hat{M}_{\mathrm{d}})_n = 0$ for all $n > 0$, which indeed yields an algebra morphism in $\mathcal{C}^{\mathrm{I_c}}$ and is therefore unique by assumption. ■

In the case of first order differential calculi the result of Proposition 4.4 can be derived under the weaker condition that $X_0 \otimes \mathrm{id}_{\mathcal{C}}$ is right exact. Before we will give the definition of left, right and bi-covariant differential calculi in $\otimes$-factor braided abelian categories, we state a corollary of the previous results concerning the extendibility of algebra morphisms to differential algebra morphisms.

Corollary 4.5 *Suppose that $(\hat{X}, \hat{\mathrm{d}}_X)$ is a differential calculus and $(\hat{Y}, \hat{\mathrm{d}}_Y)$ is an algebra in $\mathcal{C}^{\mathrm{I_c}}$. If $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ is an algebra morphism in $\mathcal{C}^{\mathrm{I}}$ such that $f_1 \circ \mathrm{d}_{X,0} = \mathrm{d}_{Y,0} \circ f_0$ then $\hat{f} : (\hat{X}, \hat{\mathrm{d}}_X) \longrightarrow (\hat{Y}, \hat{\mathrm{d}}_Y)$ is an algebra morphism in $\mathcal{C}^{\mathrm{I_c}}$.* ■

We will introduce the maximal differential calculus of a given differential algebra. It will be used in the last section of Chapter 4 for the construction of the braided exterior Hopf algebra of differential forms. Guided by Proposition 4.2 we give the following definition.

Definition 4.6 *Let $(\hat{A}, \hat{\mathrm{m}}, \hat{\eta}, \hat{\mathrm{d}})$ be an algebra in $\mathcal{C}^{\mathrm{I_c}}$. The maximal differential calculus $\hat{A}^{\mathrm{Diff}}$ is the graded subobject of $\hat{A}$ given by the inclusion $\hat{\mathrm{i}} : \hat{A}^{\mathrm{Diff}} \hookrightarrow \hat{A}$ where $\mathrm{i}_0 := \mathrm{id}_{A_0}$, $\mathrm{i}_1 := A_0\langle\mathrm{d}_0\rangle$ and $\mathrm{i}_n := \mathrm{im}(\mathrm{m}^{(n-1)}_{1\dots,1} \circ \mathrm{i}_1^{\otimes,n})$ for $n > 1$.*

The nomenclature is justified by the following proposition which can be proven easily with the help of the previous results and techniques already provided in the article.

Proposition 4.7 *On $\hat{A}^{\mathrm{Diff}}$ a differential $\hat{\mathrm{d}}^{\mathrm{Diff}}$ can be defined such that $\hat{\mathrm{i}} : (\hat{A}^{\mathrm{Diff}}, \hat{\mathrm{d}}^{\mathrm{Diff}}) \hookrightarrow (\hat{A}, \hat{\mathrm{d}})$ is an algebra morphism in $\mathcal{C}^{\mathrm{I}_c}$ and $(\hat{A}^{\mathrm{Diff}}, \hat{\mathrm{d}}^{\mathrm{Diff}})$ is a differential calculus. If $(\hat{A}', \hat{\mathrm{d}}')$ is a differential calculus and $\hat{\mathrm{j}} : (\hat{A}', \hat{\mathrm{d}}') \hookrightarrow (\hat{A}, \hat{\mathrm{d}})$ is an algebra embedding in $\mathcal{C}^{\mathrm{I}_c}$ then there exists a unique embedding $\hat{\mathrm{i}}' : (\hat{A}', \hat{\mathrm{d}}') \hookrightarrow (\hat{A}^{\mathrm{Diff}}, \hat{\mathrm{d}}^{\mathrm{Diff}})$ such that $\hat{\mathrm{j}} = \hat{\mathrm{i}} \circ \hat{\mathrm{i}}'$. If in addition $A'_0 = A_0$ then $(\hat{A}', \hat{\mathrm{d}}') \cong (\hat{A}^{\mathrm{Diff}}, \hat{\mathrm{d}}^{\mathrm{Diff}})$.* ■

Now let us mimick Woronowicz's approach and define $\hat{B}$-covariant differential calculi for graded bialgebras $\hat{B}$ in $\mathcal{C}^{\mathrm{I}}$.

Definition 4.8 *Let $\hat{B}$ be a bialgebra in the category $\mathcal{C}^{\mathrm{I}}$ and $(\hat{X}, \hat{\mathrm{d}})$ be an object in $\mathcal{C}^{\mathrm{I}_c}$. Suppose that $\hat{B} \otimes \mathrm{id}_{\mathcal{C}^{\mathrm{I}}}$ is left exact. Then $(\hat{X}, \hat{\mathrm{d}})$ is called $\hat{B}$-left covariant, $\hat{B}$-right covariant or $\hat{B}$-bicovariant differential calculus if it is a differential calculus in the category of $\hat{B}$-comodules ${}^{\hat{B}}(\mathcal{C}^{\mathrm{I}_c})$, $(\mathcal{C}^{\mathrm{I}_c})^{\hat{B}}$ and ${}^{\hat{B}}(\mathcal{C}^{\mathrm{I}_c})^{\hat{B}}$ respectively. The corresponding categories are denoted by* $\mathrm{Diff}\text{-}{}^{\hat{B}}(\mathcal{C}^{\mathrm{I}_c})$, $\mathrm{Diff}\text{-}(\mathcal{C}^{\mathrm{I}_c})^{\hat{B}}$ *and* $\mathrm{Diff}\text{-}{}^{\hat{B}}(\mathcal{C}^{\mathrm{I}_c})^{\hat{B}}$ *respectively.*

In the remainder of this chapter we are especially interested in H-bicovariant differential calculi of a bi- or Hopf algebra H in $\mathcal{C}$. We call them bicovariant differential calculi *over* H if their 0th component coincides with H. We also consider differential calculi which are bi- or Hopf algebras in $\mathcal{C}^{\mathrm{I}_c}$. They are called differential bi- or Hopf algebra calculi henceforth.

Proposition 4.9 *Given a differential bialgebra calculus $(\hat{X}, \hat{\mathrm{d}})$ in $\mathcal{C}^{\mathrm{I}_c}$. Suppose that $X_0 \otimes \mathrm{id}_{\mathcal{C}}$ is left exact. Then $(\hat{X}, \hat{\mathrm{d}})$ is a bicovariant differential calculus over X_0. Conversely, if $\mathrm{I} = \mathbf{2}$ and if (X_0, X_1, d) is a bicovariant differential calculus over X_0 then it is a differential bialgebra calculus. If X_0 is a Hopf algebra then (X_0, X_1, d) is a differential Hopf algebra calculus.*

Proof. We use the results of Proposition A.4, especially (A.3). From Proposition 3.6 one derives that X_0 is a bialgebra and X_n are X_0-Hopf bimodules for all $n \in \mathrm{I}$. This means that $(\hat{X}, \hat{\mathrm{d}})$ is an algebra which is at the same time X_0-bicomodule in $\mathcal{C}^{\mathrm{I}}$. Proposition A.1 ensures that the image condition of the differential calculus $(\hat{X}, \hat{\mathrm{d}})$ in $\mathcal{C}^{\mathrm{I}}$ can be taken over to ${}^{X_0}(\mathcal{C}^{\mathrm{I}_c})^{X_0}$. Hence $(\hat{X}, \hat{\mathrm{d}})$ is an X_0-bicovariant differential calculus. Conversely, suppose that (X_0, X_1, d) is a bicovariant differential calculus over X_0. Then (X_0, X_1, d) is a differential calculus and an X_0-bicomodule algeba in $\mathcal{C}^{\mathbf{2}_c}$. In particular X_0 is a bialgebra and X_1 is an X_0-Hopf bimodule. As a consequence of Remark 9, (X_0, X_1, d) is a differential bialgebra calculus in $\mathcal{C}^{\mathbf{2}_c}$. ■

Hence Definition 4.8 generalizes the notations of [Wor] because we know from Proposition 4.9 that a differential bialgebra calculus $(\hat{X}, \hat{\mathrm{d}})$ in $\mathcal{C}^{\mathrm{I}_c}$ is a braided bicovariant differential calculus over X_0. In the case of $\mathrm{I} = \mathbf{2}$ this is just a braided version of the definition of bicovariant differential calculi over the bialgebra X_0 given in [Wor].

Proposition 4.10 *Suppose that $\hat{A}$ is a bialgebra in $\mathcal{C}^{\mathrm{I}}$ and $(\hat{A}, \hat{\mathrm{d}})$ is an algebra in $\mathcal{C}^{\mathrm{I}_c}$ which is generated by its 0th and 1st component. If the identity*

$$\Delta_{n+1} \circ \mathrm{d}_n = \mathrm{d}_n^{\otimes} \circ \Delta_n \ : \ A_n \longrightarrow (\hat{A} \otimes \hat{A})_n \tag{4.5}$$

holds for $n=0,1$ then it is true for any $n\in\mathrm{I}$ and therefore $\hat{A}$ is bialgebra in $\mathcal{C}^{\mathrm{I}_c}$. If in addition (A_0,A_1,d_0) is a bialgebra in $\mathcal{C}^{\mathbf{2}_c}$ and $\hat{A}^{\mathrm{Diff}}\hookrightarrow\hat{A}$ is a sub-bialgebra in $\mathcal{C}^{\mathrm{I}}$ then $(\hat{A}^{Diff},\hat{\mathrm{d}}^{Diff})$ is bialgebra in $\mathcal{C}^{\mathrm{I}_c}$. In particular it is a bicovariant differential calculus over A_0 if $A_0\otimes\mathrm{id}_{\mathcal{C}}$ is left exact.

Proof. To prove the first part we only have to consider $\mathrm{I}=\underline{\mathbb{N}}$. Suppose that (4.5) holds for $n\geq 1$. Then

$$\begin{aligned}(\Delta_{n+2}\circ\mathrm{d}_{n+1})_{k,l}\circ\mathrm{m}_{1,n} &= (\Delta_{n+2}\circ\mathrm{d}_{n+1}\circ\mathrm{m}_{1,n})_{k,l;1,n}\\ &= (\mathrm{m}^{\otimes}_{n+2}\circ(\Delta\otimes\Delta)_{n+2}\circ\mathrm{d}^{\otimes}_{n+1})_{k,l;1,n}\\ &= (\mathrm{m}^{\otimes}_{n+2}\circ\mathrm{d}^{\otimes^2}_{n+1}\circ(\Delta\otimes\Delta)_{n+1})_{k,l;1,n}\\ &= (\mathrm{d}^{\otimes}_{n+1}\circ\Delta_{n+1}\circ\mathrm{m}_{n+1})_{k,l;1,n} = (\mathrm{d}^{\otimes}_{n+1}\circ\Delta_{n+1})_{k,l}\circ\mathrm{m}_{1,n}\,.\end{aligned}$$

In the third equation use has been made of the induction hypothesis since $1,n<n+1$. Therefore (4.5) holds for $n+1$, because $\mathrm{m}_{1,n}$ is supposed to be an epimorphism.

For the proof of the second part we note that

$$\begin{aligned}\Delta_2\circ\mathrm{d}_1\circ\mathrm{m}_{0,1}\circ(\mathrm{id}_A\otimes\mathrm{d}_0) &= \Delta_2\circ\mathrm{m}_{1,1}\circ(\mathrm{d}_0\otimes\mathrm{d}_0)\\ &= \mathrm{m}^{\otimes}_{1,1}\circ(\Delta_1\circ\mathrm{d}_0\otimes\Delta_1\circ\mathrm{d}_0)\\ &= \mathrm{m}^{\otimes}_{1,1}\circ(\mathrm{d}^{\otimes}_0\otimes\mathrm{d}^{\otimes}_0)\circ(\Delta_0\otimes\Delta_0)\\ &= \mathrm{d}^{\otimes}\circ\mathrm{m}^{\otimes}_{0,1}\circ(\Delta_0\otimes\Delta_1\circ\mathrm{d}_0)\\ &= \mathrm{d}^{\otimes}\circ\Delta_1\circ\mathrm{m}_{0,1}\circ(\mathrm{id}_A\otimes\mathrm{d}_0)\,,\end{aligned}$$

from which follows $\Delta^{\mathrm{Diff}}_2\circ\mathrm{d}^{\mathrm{Diff}}_1=\mathrm{d}^{\mathrm{Diff}\otimes}_1\circ\Delta^{\mathrm{Diff}}_1$. Since $(A_0,A_1,\mathrm{d}_0,\Delta_0,\Delta_1)$ is a bialgebra in $\mathcal{C}^{\mathbf{2}_c}$ the analogous relation will be decuced for the 0th component. Then we can apply the first part of the proposition to finish the proof. ∎

The following proposition is important for the subsequent investigation as well as for the construction of the braided exterior Hopf algebra of differential forms in the next section.

Proposition 4.11 *Let $\mathrm{I}=\mathbf{2}$ or $\mathrm{I}=\underline{\mathbb{N}}$ and $\mathcal{C}$ be a $\otimes$-factor braided monoidal category. Suppose that $\hat{A}$ is an algebra in $\mathcal{C}^{\mathrm{I}}$ and $x:\mathbf{1}\longrightarrow A_1$ is a morphism in $\mathcal{C}$ such that $\mathrm{m}_{2,n}\circ\big(\mathrm{m}_{1,1}\circ(x\otimes x)\otimes\mathrm{id}_n\big)=\mathrm{m}_{n,2}\circ\big(\mathrm{id}_n\otimes\mathrm{m}_{1,1}\circ(x\otimes x)\big)$ for all $n\in\mathrm{I}$ if $\mathrm{I}=\underline{\mathbb{N}}$. Then the morphisms*

$$[x,\cdot]_n:=\mu_{1,n}\circ(x\otimes\mathrm{id}_{A_n})-(-1)^n\mu_{n,1}\circ(\mathrm{id}_{A_n}\otimes x):A_n\longrightarrow A_{n+1}\tag{4.6}$$

for any $n,n+1\in\mathrm{I}$, turn $(\hat{A},[x,\cdot]^\wedge)$ into an algebra in $\mathcal{C}^{\mathrm{I}_c}$.

If $(\hat{A},\hat{\Delta},\hat{\varepsilon})$ is a bialgebra in $\mathcal{C}^{\mathrm{I}}$, x is bi-invariant, i. e. $\Delta_{0,1}\circ x=\eta\otimes x$ and $\Delta_{1,0}\circ x=x\otimes\eta$, and if the maximal differential calculus $\hat{A}^{Diff}$ of $(\hat{A},[x,\cdot]^\wedge)$ is a sub-bialgebra of $\hat{A}$ in $\mathcal{C}^{\mathrm{I}}$, then $(\hat{A}^{Diff},([x,\cdot]^\wedge)^{Diff})$ is a bialgebra in $\mathcal{C}^{\mathrm{I}_c}$.

Proof. The following identities prove the differential algebra property for $(\hat{A},[x,\cdot]^\wedge)$.

$$\begin{aligned}[x,\cdot]_{m+n}\circ\mathrm{m}_{m,n} &= \mathrm{m}_{1,m+n}\circ(x\otimes\mathrm{m}_{m,n})+(-1)^{m+n+1}\mathrm{m}_{m+n,1}\circ(\mathrm{m}_{m,n}\otimes x)\\ &= \mathrm{m}_{1,m,n}\circ(x\otimes\mathrm{id}_{A_m\otimes A_n})+(-1)^{m+n+1}\mathrm{m}_{m,n,1}\circ(\mathrm{id}_{A_m\otimes A_n}\otimes x)\\ &= \mathrm{m}_{m+1,n}\circ\Big(\big(\mathrm{m}_{1,m}\circ(x\otimes\mathrm{id}_{A_m})-(-1)^m\mathrm{m}_{m,1}\circ(\mathrm{id}_{A_m}\otimes x)\big)\otimes\mathrm{id}_{A_n}\Big)\\ &\quad+(-1)^m\mathrm{m}_{m,n+1}\circ\Big(\mathrm{id}_{A_m}\otimes\big(\mathrm{m}_{1,n}\circ(x\otimes\mathrm{id}_{A_n})-(-1)^n\mathrm{m}_{n,1}\circ(\mathrm{id}_{A_n}\otimes x)\big)\Big)\\ &= \mathrm{m}_{m+1,n}\circ([x,\cdot]_m\otimes\mathrm{id}_{A_n})+(-1)^m\mathrm{m}_{m,n+1}\circ(\mathrm{id}_{A_m}\otimes[x,\cdot]_n)\end{aligned}$$

and

$$[x,\cdot]_{n+1}\circ[x,\cdot]_n = \mathrm{m}_{1,1,n}\circ(x\otimes x\otimes \mathrm{id}_{A_n}) + (-1)^{n+1}\mathrm{m}_{1,n,1}\circ(x\otimes \mathrm{id}_{A_n}\otimes x) + \\ + (-1)^{n+2}\mathrm{m}_{1,n,1}\circ(x\otimes \mathrm{id}_{A_n}\otimes x) + -\mathrm{m}_{n,1,1}\circ(\mathrm{id}_{A_n}\otimes x\otimes x) = 0\,.$$

The assumption that $\hat{A}$ is a bialgebra in $\mathcal{C}^{\mathrm{I}}$ and the bi-invariance property of x according to the second part of the proposition verify directly that $\big((A_0,A_1),[x,\cdot]_0,\Delta_0,\Delta_1\big)$ is a bialgebra in $\mathcal{C}^{\mathbf{2}_c}$. The application of Proposition 4.10 then completes the proof. ■

In the remainder of this section we will look in more detail to first order differential calculi providing the corresponding facts which are necessary for the understanding of the final section on exterior differential Hopf algebras. We will begin with the study of the category of derivations $\mathrm{Der}(A)$ over an algebra A in $\mathcal{C}$.

Definition 4.12 *Let A be an algebra in $\mathcal{C}$. Then the category* $\mathrm{Der}(A)$ *of derivations over A consists of objects (A,X,d_X) which are algebras in $\mathcal{C}^{\mathrm{I}_c}$, and of algebra morphisms in $\mathcal{C}^{\mathrm{I}_c}$ of the form* $(\mathrm{id}_A,f):(A,X,\mathrm{d}_X)\longrightarrow(A,Y,\mathrm{d}_Y)$.

A morphism in $\mathrm{Der}(A)$ can therefore equivalently be seen as an A-bimodule morphism $f: X\longrightarrow Y$ such that $f\circ \mathrm{d}_X = \mathrm{d}_Y$. As a side-step the following proposition and corollary will be derived using derivations. The statements are braided analogues of the corresponding results of [Wor].

Proposition 4.13 *Given an algebra (A,m,η) in $\mathcal{C}$. We consider $A_{(2)} := \mathrm{Ker}(\mathrm{m})$ and the unique morphism $D_A: A\longrightarrow A_{(2)}$ such that $\mathrm{ker}(\mathrm{m})\circ D_A = \eta\otimes\mathrm{id}_A - \mathrm{id}_A\otimes\eta$. Then $(A,A_{(2)},D_A)$ is a first order differential calculus. It is an initial object in $\mathrm{Der}(A)$. If B is a bialgebra in $\mathcal{C}$ with left exact functor $B\otimes\mathrm{id}_{\mathcal{C}}$ then $(B,B_{(2)},D_B)$ is a bicovariant differential calculus over B.*

Proof. One can easily verify within the category $\mathrm{Der}(A)$ that

$$(A,A_{(2)},D_A) = \mathrm{Ker}\big\{(A,A\otimes A,\eta\otimes\mathrm{id}_A - \mathrm{id}_A\otimes\eta)\xrightarrow{(\mathrm{id}_A,\mathrm{m})}(A,A,0)\big\} \qquad \in \mathrm{Ob}(\mathrm{Der}(A))\,.$$

The multiplication m is an epimorphism and therefore $\mathrm{m} = \mathrm{coker}(\mathrm{id}_{A\otimes A} - \mathrm{m}\otimes\eta)$ because $f\circ(\mathrm{id}_{A\otimes A}-\mathrm{m}\otimes\eta)=0$ implies that f factorizes over m in the form $f = f\circ(\mathrm{id}\otimes\eta)\circ\mathrm{m}$. So $A\langle D_A\rangle = \mathrm{Im}(\mathrm{id}_{A\otimes A}-\mathrm{m}\otimes\eta) = \mathrm{Ker}\,\mathrm{m} = A_{(2)}$ and $(A,A_{(2)},D_A)$ is a differential calculus. By Proposition 4.3 there is at most one morphism in $\mathrm{Der}(A)$ from the differential calculus $(A,A_{(2)},D_A)$ to any other object (A,X,d_X). We have

$$\big(\mu_\ell^X\circ(\mathrm{id}\otimes\mathrm{d}_X)+\mu_r^X\circ(\mathrm{d}_X\otimes\mathrm{id})\big)\circ\mathrm{ker}\,\mathrm{m} = \mathrm{d}_X\circ\mathrm{m}\circ\mathrm{ker}\,\mathrm{m} = 0\,.$$

Denoting $\pi := \mu_\ell^X\circ(\mathrm{id}\otimes\mathrm{d}_X)\circ\mathrm{ker}\,\mathrm{m} = -\mu_r^X\circ(\mathrm{d}_X\otimes\mathrm{id})\circ\mathrm{ker}\,\mathrm{m}$ one can directly verify that $(\mathrm{id}_A,\pi)\,:\,(A,A_{(2)},D_A)$ $\longrightarrow(A,X,\mathrm{d}_X)$ is a morphism in $\mathrm{Der}(A)$. ■

In a similar manner as in [Wor] we can describe bicovariant first order differential calculi over a Hopf algebra H as certain sub-bimodules of $H_{(2)}$. The corresponding result can be derived as a corollary of Proposition 4.13 using Lemma A.5.

Corollary 4.14 *The bicovariant first order differential calculi over a flat Hopf algebra H with isomorphic antipode in $\mathcal{C}$ are in one-to-one correspondence with either*

1. *Hopf sub-bimodules of $H_{(2)}$,*
2. *crossed sub-bimodules of* $\ker \varepsilon$

where the counit ε is regarded as crossed module morphism $\varepsilon : H^{\mathrm{ad}} \longrightarrow \mathbf{1}$.

Proof. Let A be any algebra in $\mathcal{C}$ such that $A \otimes \mathrm{id}_{\mathcal{C}}$ is right exact. Let (A, X, d_X) be a differential calculus and $(\mathrm{id}_A, f) : (A, X, \mathrm{d}_X) \longrightarrow (A, A_{(2)}, D_A)$ be a morphism in $\mathrm{Der}(A)$. Then f is an epimorphism because $f \circ \mathrm{d}_X = D_A$ and $(A, A_{(2)}, D_A)$ is a differential calculus. Conversely, if $f : A_{(2)} \longrightarrow X$ is a bimodule epimorphism then $(A, X, f \circ D_A)$ is a differential calculus. So first order differential calculi over A are in one-to-one correspondence with sub-bimodules $\mathrm{Ker}\, f$ of $A_{(2)}$. One can specify this result to the Hopf algebra H which is an algebra in ${}^H\mathcal{C}^H$ through its comultiplication. Then the first statement of the corollary is derived.

Lemma A.5 and the equivalence ${}^H_H\mathcal{C}^H_H \simeq \mathcal{DY}(\mathcal{C})^H_H$ from Theorem 1.4 imply the correspondence between $H_{(2)} \in \mathrm{Ob}\big({}^H_H\mathcal{C}^H_H\big)$ and $\mathrm{Ker}\,\varepsilon \in \mathrm{Ob}\big(\mathcal{DY}(\mathcal{C})^H_H\big)$, and therefore between Hopf sub-bimodules of $H_{(2)}$ and crossed sub-bimodules of $\mathrm{Ker}\,\varepsilon$. ∎

For our further investigations it is convenient to work with the notion of a comma category in the simplest case. Its definition is outlined in the Appendix. In Proposition 4.19 the comma category $(\mathbf{1} \downarrow \mathcal{C})$ will be used to formulate the braided Woronowicz construction of a Hopf bimodule with bi-invariant element out of a first order bicovariant differential calculus. Before, we derive some results relating bimodules in $(\mathbf{1} \downarrow \mathcal{C})$ and derivations.

Lemma 4.15 *Let (A, m, η) be an algebra in $\mathcal{C}$ then $\big((A, \eta), \mathrm{m}, \eta\big)$ is an algebra in $(\mathbf{1} \downarrow \mathcal{C})$. An analogous result holds for bi- and Hopf algebras. A module (X, μ) over the algebra A in $\mathcal{C}$ for which (X, x) is an object in $(\mathbf{1} \downarrow \mathcal{C})$ is also an (A, η)-module $((X, x), \mu)$ in $(\mathbf{1} \downarrow \mathcal{C})$.*

Proof. Of course, trivially $\mathrm{m} \circ (\eta \otimes \eta) = \eta$, $\eta \circ \mathrm{id}_{\mathbf{1}} = \eta$ and similarly for the comultiplication, counit and antipode of bi- and Hopf algebras. Therefore $\big((A, \eta), \mathrm{m}, \eta\big)$ is an algebra in $(\mathbf{1} \downarrow \mathcal{C})$. Analogously the equation $\mu_l \circ (\eta \otimes x) = x$ shows that $\big((X, x), \mu_l\big)$ is an (A, η)-left module in $(\mathbf{1} \downarrow \mathcal{C})$. ∎

Restricting to $\mathrm{I} = \mathbf{2}$ one finds a similar result as in [Wor] which is converse to Proposition 4.11.

Proposition 4.16 *Given a derivation (X, d) over an algebra A in $\mathcal{C}$. Then an (A, η)-bimodule $A \oplus_{\mathrm{d}} X := \big(\big(A \oplus X, \binom{\eta}{0}\big), \mu_{\mathrm{d},l}, \mu_{\mathrm{d},r}\big)$ in $(\mathbf{1} \downarrow \mathcal{C})$ can be constructed where the modul actions are given through*

$$\mu_{\mathrm{d},l} = \begin{pmatrix} \mathrm{m} & 0 \\ 0 & \mu_l \end{pmatrix} \quad \text{and} \quad \mu_{\mathrm{d},r} = \begin{pmatrix} \mathrm{m} & 0 \\ \mu_l \circ (\mathrm{id}_A \otimes \mathrm{d}) & \mu_r \end{pmatrix} . \tag{4.7}$$

Therefore in particular $\big(A, A \oplus_{\mathrm{d}} X, [\binom{\eta}{0}, \cdot]\big)$ is an algebra in $\mathcal{C}^{\mathbf{2}_c}$ according to Proposition 4.11. Moreover $\big[\binom{\eta}{0}, \cdot\big] = \binom{0}{\mathrm{d}}$.

PROOF. For as the proof of the proposition does not require difficult techniques we exemplarily outline that $\mu_{\mathrm{d},r}$ is a right module action. It holds

$$\mu_{\mathrm{d},r} \circ (\mathrm{id}_{A \oplus_{\mathrm{d}} X} \otimes \eta) = \begin{pmatrix} \mathrm{m} & 0 \\ \mu_l \circ (\mathrm{id}_A \otimes \mathrm{d}) & \mu_r \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_A \otimes \eta & 0 \\ 0 & \mathrm{id}_X \otimes \eta \end{pmatrix} = \mathrm{id}_{A \oplus_{\mathrm{d}} X} \tag{4.8}$$

and

$$\begin{aligned} \mu_{\mathrm{d},r} \circ (\mu_{\mathrm{d},r} \otimes \mathrm{id}_A) &= \begin{pmatrix} \mathrm{m} & 0 \\ \mu_l \circ (\mathrm{id}_A \otimes \mathrm{d}) & \mu_r \end{pmatrix} \circ \begin{pmatrix} \mathrm{m} \otimes \mathrm{id}_A & 0 \\ \mu_l \circ (\mathrm{id}_A \otimes \mathrm{d}) \otimes \mathrm{id}_A & \mu_r \otimes \mathrm{id}_A \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{m} \circ (\mathrm{id}_A \otimes \mathrm{m}) & 0 \\ \mu_l \circ (\mathrm{id}_A \otimes \mathrm{d} \circ \mathrm{m}) & \mu_r \circ (\mathrm{id}_X \otimes \mathrm{m}) \end{pmatrix} \\ &= \mu_{\mathrm{d},r} \circ (\mathrm{id}_{A \oplus_{\mathrm{d}} X} \otimes \mathrm{m}) \,. \end{aligned} \tag{4.9}$$

The equations (4.8) and (4.9) prove the statement. The relation $\mu_{\mathrm{d},r} \circ \left(\binom{\eta}{0} \otimes \eta\right) = \binom{\eta}{0}$, which is a consequence of (4.8), implies that $\mu_{\mathrm{d},r}$ is a morphism in $(\mathbf{1} \downarrow \mathcal{C})$. ∎

The results of Propositions 4.11 and 4.16 are used to formulate the following theorem which shows that the categories ${}_{(A,\eta)}(\mathbf{1} \downarrow \mathcal{C})_{(A,\eta)}$ and $\mathrm{Der}(A)$ are mutually adjoint.

THEOREM 4.17 *Let A be an algebra in $\mathcal{C}$. Then the functors*

$$\mathrm{Der}(A) \underset{[\_]_A}{\overset{A \oplus_{\mathrm{d}} (\_)}{\rightleftarrows}} {}_{(A,\eta)}(\mathbf{1} \downarrow \mathcal{C})_{(A,\eta)} \tag{4.10}$$

are defined on the objects by $A \oplus_{\mathrm{d}} (X, \mathrm{d}) = A \oplus_{\mathrm{d}} X$ and $[(X, x)]_A = (X, [x, \cdot])$ and on morphisms by $A \oplus_{\mathrm{d}} (f) = \mathrm{id}_A \oplus f$ and $[g]_A = g$ respectively. The functor $A \oplus_{\mathrm{d}} (\_)$ is left adjoint of the functor $[\_]_A$ through the (inverse of the) natural bijection $\Gamma_{(X,\mathrm{d}),(Y,y)} : \mathrm{Hom}_{\mathrm{Der}(A)}\big((X, \mathrm{d}), (Y, [y, \cdot])\big) \longrightarrow \mathrm{Hom}_{{}_{(A,\eta)}(\mathbf{1}\downarrow\mathcal{C})_{(A,\eta)}}\big(A \oplus_{\mathrm{d}} X, (Y, y)\big)$, $f \mapsto \big(\mu_l^Y \circ (\mathrm{id}_A \otimes y), f\big)$.

PROOF. Straightforward calculations like in Proposition 4.16 show that $\big(\mu_l^Y \circ (\mathrm{id}_A \otimes y), f\big)$ is an A-bimodule morphism. Moreover, for any $(g, f) \in \mathrm{Hom}_{{}_{(A,\eta)}(\mathbf{1}\downarrow\mathcal{C})_{(A,\eta)}}\big(A \oplus_{\mathrm{d}} X, (Y, y)\big)$ the condition $(g, f) \circ \binom{\eta}{0} = y$ is equivalent to $g = \mu_l^Y \circ (\mathrm{id}_A \otimes y)$. ∎

LEMMA 4.18 *Suppose that B is a bialgebra in $\mathcal{C}$ with left exact functor $B \otimes \mathrm{id}_{\mathcal{C}}$. The full subcategory of* $\mathrm{Der}(B)$ *generated by left, right or bicovariant first order differential calculi over B is a subcategory of the left, right or bicovariant first order differential calculi over B respectively. We denote it by* $\mathrm{L}_{\mathcal{C}}$-$\mathrm{Der}(B)$, $\mathrm{R}_{\mathcal{C}}$-$\mathrm{Der}(B)$ *and* $\mathrm{Bi}_{\mathcal{C}}$-$\mathrm{Der}(B)$ *respectively.*

PROOF. It only has to be proved that the morphisms are (bi-)comodule morphisms. We demonstrate the case of left comodules. Suppose that $f : (X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{d}_Y)$ is a morphism in $\mathrm{L}_{\mathcal{C}}$-$\mathrm{Der}(B)$. Then the relations in Figure 3 are fulfilled. In the first equation of Figure 3 we used that f is a B-bimodule morphism and the identity $f \circ \mathrm{d}_X = \mathrm{d}_Y$. The second equality is obtained by applying the obvious Hopf module properties of a left covariant differential calculus. With the same techniques as exploited in the first and second equations we derive the equalities three and four. Since by assumption $\mu_l^X \circ (\mathrm{id}_B \otimes \mathrm{d}_X)$ is an epimorphism we conclude from Figure 3 that f is a B-left comodule morphism. ∎

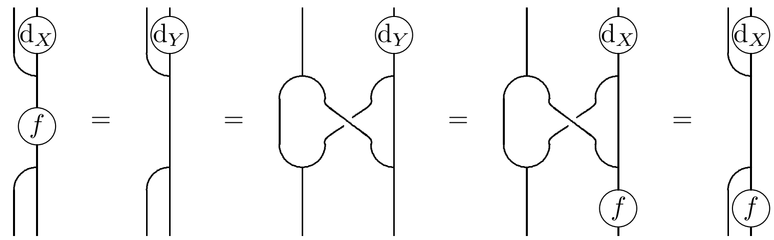

Figure 3: Proof of the comodule morphism property.

Finally Lemma 4.18 yields the proposition which is a formal Woronowicz construction of Hopf bimodules with invariant elements.

Proposition 4.19 *Let H be a flat Hopf algebra with isomorphic antipode in $\mathcal{C}$. Then the following functors can be defined canonically.*

$$\begin{aligned} H \oplus_{\mathrm{d}} (\_) : \mathrm{L}_{\mathcal{C}}\text{-Der}(H) &\longrightarrow {}^{(H,\eta)}_{(H,\eta)}(\mathbf{1} \downarrow \mathcal{C})_{(H,\eta)}\,, \\ H \oplus_{\mathrm{d}} (\_) : \mathrm{R}_{\mathcal{C}}\text{-Der}(H) &\longrightarrow {}_{(H,\eta)}(\mathbf{1} \downarrow \mathcal{C})^{(H,\eta)}_{(H,\eta)}\,, \\ H \oplus_{\mathrm{d}} (\_) : \mathrm{Bi}_{\mathcal{C}}\text{-Der}(H) &\longrightarrow {}^{(H,\eta)}_{(H,\eta)}(\mathbf{1} \downarrow \mathcal{C})^{(H,\eta)}_{(H,\eta)} \end{aligned} \tag{4.11}$$

where the coactions on the objects $H \oplus_{\mathrm{d}} (X, \mathrm{d}_X)$ are given by $\nu_{\mathrm{d},l} = \left(\begin{smallmatrix} \Delta & 0 \\ 0 & \nu_l \end{smallmatrix}\right)$ and $\nu_{\mathrm{d},r} = \left(\begin{smallmatrix} \Delta & 0 \\ 0 & \nu_r \end{smallmatrix}\right)$.

Proof. With the help of Proposition 4.11, Theorem 4.17 and Lemma 4.18 the proof of the proposition follows straightforwardly. ■

Remark 16 In particular for a bicovariant differential calculus (X, d) over H this means that $\binom{\eta}{0} : \mathbf{1} \longrightarrow H \oplus_{\mathrm{d}} X$ is bi-invariant. I. e. $\nu_l \circ \binom{\eta}{0} = \eta \otimes \binom{\eta}{0}$ and $\nu_r \circ \binom{\eta}{0} = \binom{\eta}{0} \otimes \eta$.

Remark 17 We have been working with Hopf algebras which have an isomorphic antipode in $\mathcal{C}$. This condition could be omitted at several places. Then instead one would obtain *pseudo-braided* categories of Hopf bimodules. Pseudo-braided categories are pre-braided categories where the pre-braiding is an isomorphism if one of the tensor factors is the unit object. Many results obtained in the article could be transferred to this case without difficulties.

Exterior Differential Hopf Algebras

In this section we construct the braided exterior Hopf algebra of differential forms of a given first order bicovariant differential calculus. This object is a differential Hopf algebra calculus and extends the initial first order differential calculus uniquely.

In Lemma 4.20 we build the differential algebra $\left((H \oplus_{\mathrm{d}} X)^{\wedge_H}, \hat{\mathrm{D}}\right)$ out of the bicovariant differential calculus (X, d) over H according to Propositions 4.19, 4.11 and Definition 3.14. Then in Theorem 4.21 the exterior algebra $(X^{\wedge_H}, \mathrm{d}^{\wedge})$ is derived as the maximal calculus of $\left((H \oplus_{\mathrm{d}} X)^{\wedge_H}, \hat{\mathrm{D}}\right)$.

Lemma 4.20 *Suppose that (X, d) is a first order bicovariant differential calculus over the flat Hopf algebra H with isomorphic antipode in the category $\mathcal{C}$. Then the graded Hopf algebra $(H \oplus_{\mathrm{d}} X)^{\wedge_H}$ is a differential algebra in $\mathcal{C}^{\underline{\mathbb{N}}_c}$ through the differential $\hat{\mathrm{D}} := [\binom{\eta}{0}, \cdot] = \binom{0}{\mathrm{d}}$ which is derived from $\binom{\eta}{0} : \mathbf{1} \longrightarrow H \oplus X$ according to Proposition 4.11.*

Proof. We have to prove that $\mathrm{m}_{2,n} \circ (\mathrm{m}_{1,1} \circ (x \otimes x) \otimes \mathrm{id}_n) = \mathrm{m}_{n,2} \circ (\mathrm{id}_n \otimes \mathrm{m}_{1,1} \circ (x \otimes x))$ for all $n \in \mathrm{I}$, where $\mathrm{m} := \mathrm{m}^{(H \oplus_{\mathrm{d}} X)^{\wedge_H}}$ and $x := \binom{\eta}{0}$. Then all the conditions of the first part of Proposition 4.11 are satisfied.

We are using Theorem 1.5 and the relation corresponding to (3.35) involving the multiplication $\mathrm{m}^{(H \oplus_{\mathrm{d}} X)^{\wedge_H}}$. Then we derive

$$\mathrm{m}_{1,1}^{(H \oplus_{\mathrm{d}} X)^{\wedge_H}} \circ \left(\begin{pmatrix} \eta \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right) = \mathrm{coim}(\hat{A})_2 \circ \left(\begin{pmatrix} \eta \\ 0 \end{pmatrix} \otimes {}_{H \oplus_{\mathrm{d}} X}\mathrm{p} \circ \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right) = 0 \tag{4.12}$$

because

$$\begin{gathered} {}^{H}_{H}\mathcal{C}^{H}_{H}\Psi_{H \oplus_{\mathrm{d}} X, H \oplus_{\mathrm{d}} X} \circ \left(\begin{pmatrix} \eta \\ 0 \end{pmatrix} \otimes {}_{H \oplus_{\mathrm{d}} X}\mathrm{p} \circ \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} \eta \\ 0 \end{pmatrix} \otimes {}_{H \oplus_{\mathrm{d}} X}\mathrm{p} \circ \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right), \\ (\hat{A}_{H \oplus_{\mathrm{d}} X})_2 = \mathrm{id}_{\mathcal{T}^H_{\mathcal{C}}(H \oplus_{\mathrm{d}} X)_2} - {}^{H}_{H}\mathcal{C}^{H}_{H}\Psi_{H \oplus_{\mathrm{d}} X, H \oplus_{\mathrm{d}} X} . \end{gathered} \tag{4.13}$$

This concludes the proof. ∎

Remark 18 It can be shown that the object $\left((H \oplus_{\mathrm{d}} X)^{\wedge_H}, \hat{D}\right)$ is even a differential Hopf algebra. However, this fact will not be used in the sequel. Therefore we only mention it here in the remark.

We exploit Lemma 4.20 to prove the central theorem in Chapter 4.

Theorem 4.21 *Let H be a flat Hopf algebra with isomorphic antipode in $\mathcal{C}$, and (X, d) be a first order bicovariant differential calculus over H. Then there exists a differential $\mathrm{d}^\wedge$ such that $(X^{\wedge_H}, \mathrm{d}^\wedge)$ is a differential Hopf algebra calculus in $\mathcal{C}^{\underline{\mathbb{N}}_c}$ which extends the first order calculus (X, d), i. e. $\mathrm{d}_0^\wedge = \mathrm{d}$.*

We call $(X^{\wedge_H}, \mathrm{d}^\wedge)$ the braided exterior tensor Hopf algebra of differential forms over (X, d). Explicitely, $(X^{\wedge_H}, \mathrm{d}^\wedge) = \left((H \oplus_{\mathrm{d}} X)^{\wedge_H}, \hat{D}\right)^{Diff}$.

Proof. Since $\binom{0}{\mathrm{id}_X} : X \longrightarrow H \oplus_{\mathrm{d}} X$ is a Hopf bimodule monomorphism, Proposition 3.13 yields a monomorphic Hopf algebra morphism $\hat{\mathrm{j}} : X^{\wedge_H} \longrightarrow (H \oplus_{\mathrm{d}} X)^{\wedge_H}$. According to Proposition 4.7 the 0th and the 1st component of the embedding of the maximal differential calculus into $((H \oplus_{\mathrm{d}} X)^{\wedge_H}, \hat{\mathrm{D}})$ are given by $\mathrm{i}_0 = \mathrm{id}_H$ and $\mathrm{i}_1 = \mathrm{im}(\mathrm{m}^{(H \oplus_{\mathrm{d}} X)^{\wedge_H}} \circ (\mathrm{id}_H \otimes \mathrm{D}_0)) = \binom{0}{\mathrm{im}(\mu_l \circ (\mathrm{id}_H \otimes \mathrm{d}))} = \binom{0}{\mathrm{id}_X} \cong \mathrm{j}_1$ where it has been used that the left action of H on $H \otimes_{\mathrm{d}} X$ is diagonal and that (X, d) is a differential calculus. Because $\hat{\mathrm{i}}$ and $\hat{\mathrm{j}}$ are algebra morphisms, and both $X^{\wedge_H}$ and $(H \otimes_{\mathrm{d}} X)^{\wedge_H \mathrm{Diff}}$ are generated multiplicatively by their 0th and 1st component, it follows that $\hat{\mathrm{i}} = \hat{\mathrm{j}}$. Hence $(X^{\wedge_H}, \mathrm{d}^\wedge) = \left((H \oplus_{\mathrm{d}} X)^{\wedge_H}, \hat{\mathrm{D}}\right)^{\mathrm{Diff}}$ and $\mathrm{d}_0^\wedge = \mathrm{d}$. Since $\binom{\eta}{0}$ is bi-invariant by Proposition 4.19 we can apply the Propositions 4.11 and 3.7 to finish the proof of the theorem. ∎

Using Theorem 3.17 and Corollary 4.5 the following corollary can be deduced.

Corollary 4.22 *Suppose that $(\hat{Y}, \hat{\mathrm{d}}_Y)$ is a differential Hopf algebra calculus. If $(g_0, g_1) : (Y_0, Y_1, \mathrm{d}_{Y,0}) \longrightarrow (H, X, \mathrm{d})$ is a bialgebra morphism in $\mathcal{C}^{\mathbf{2}_c}$ then there exists a unique bialgebra morphism $g^{\wedge_H} : (\hat{Y}, \hat{\mathrm{d}}_Y) \longrightarrow (X^{\wedge_H}, \mathrm{d}^\wedge)$ in $\mathcal{C}^{\underline{\mathbb{N}}_c}$ which extends (g_0, g_1).* ∎

Remark 19 In [Vla, RV] the intrinsic Hopf algebra structure of the higher order Woronowicz differential calculus over a first order bicovariant differential calculus is shown to generate very naturally a bicovariant algebra of four basic objects within a differential calculus on quantum groups as the Heisenberg double of two mutually dual graded Hopf algebras. Namely coordinate functions, differential 1-forms, Lie derivatives, and inner derivations.

This construction works completely analogous in the braided setting. The definition of the Heisenberg double can be directly generalized to the braided case if we impose some further conditions on the braided category $\mathcal{C}$ like, for instance, rigidity (see [BKLT] for such a possibility). A more detailed description of these additional properties of $\mathcal{C}$ is required for the rigorous mathematical treatment – this will be published elsewhere. Then the generalization is based on the $(((H^\vee)_{\rm op})^{\rm op})$-left module algebra structure on H. And the covariance properties of the corresponding cross product algebra $H\#(((H^\vee)_{\rm op})^{\rm op})$ are checked literally.

For dually paired Hopf algebras $\big((A,X),(A^*,X^*)\big)$ in $\mathcal{C}^{\mathbf{2}}$ the external Hopf algebras $X^\wedge$, $X^{*\wedge}$ in $\mathcal{C}^{\underline{\mathbb{N}}}$ are dually paired in a natural way. And the formal analogue of the Heisenberg double $X^\wedge\#X^{*\wedge}$ can be constructed. Let moreover (A,X,d) be a bicovariant differential calulus. Note that in this case the "Cartan identity" is a special case of the differential coalgbera property $\Delta_{1,n}\circ\mathrm{d}=(\mathrm{d}\otimes\mathrm{id})\circ\Delta_{0,n}-(\mathrm{id}\otimes\mathrm{d})\circ\Delta_{1,n-1}$. We simply have to switch the left "coaction" $\Delta_{1,\_}$ of X into a right action of the dual X^*.

The results of our article show that it is possible to generalize Woronowicz's and others' results on (higher order bicovariant) differential calculi to a very general class of braided abelian categories. The essential tools for the succesful approach are the braided Hopf bimodules and crossed modules. In rigid braided monoidal categories an appropriate definition of a (non-degenerate) pairing of objects should basically lead to versions of Woronowicz's "quantum Lie algebras" in braided categories and to a description of differential calculi in terms of "braided linear functionals". The resulting "quantum Lie algebras" may fit into the framework of Majid's braided Lie algebras [Ma7]. Once a suitable braided category is fixed the "functional" description of bicovariant differential calculi could provide another tool for attacking problems like their classification in a way similar as in [BGMST, Ma8, Ros, SS]. This will be our direction for further investigations on this subject.

A Appendix

In the appendix we set out some general categorical results and derivations of previous outcomes which will be needed in the course of the present work or are related with it.

Proposition A.1 *Let $\mathcal{C}$ be a braided abelian category with bi-additive tensor product. Let A be an algebra in $\mathcal{C}$ and $A\otimes\mathrm{id}_{\mathcal{C}}$ be right exact. Then the categories of modules ${}_A\mathcal{C}$, $\mathcal{C}_A$ and the category of bimodules ${}_A\mathcal{C}_A$ are abelian. The dual result holds for the comodules of a coalgebra C if the functor $C\otimes\mathrm{id}_{\mathcal{C}}$ is supposed to be left exact. If B is a flat bialgebra in $\mathcal{C}$, then also the several categories of Hopf (bi-)modules ${}^B_B\mathcal{C}$, $\mathcal{C}^B_B$, ${}^B_B\mathcal{C}_B$, ${}_B\mathcal{C}^B_B$,…,${}^B_B\mathcal{C}^B_B$ and $\mathcal{DY}(\mathcal{C})^B_B$ are abelian. Let H be a flat Hopf algebra with invertible antipode in $\mathcal{C}$ then the categories ${}^H_H\mathcal{C}^H_H$ and $\mathcal{DY}(\mathcal{C})^H_H$ are braided abelian with bi-additive tensor product. In this case in particular the equivalences $H\ltimes(\_):\mathcal{DY}(\mathcal{C})^H_H\leftrightarrows{}^H_H\mathcal{C}^H_H:{}_H(\_)$ of Theorem 1.4 and the several forgetful functors, especially ${}^H_H\mathcal{C}^H_H\longrightarrow\mathcal{C}$, $\mathcal{DY}(\mathcal{C})^H_H\longrightarrow\mathcal{C}$ are exact.*

Proof. We have $\mathrm{id}_A\otimes\operatorname{coker}f=\operatorname{coker}(\mathrm{id}_A\otimes f)$ when $A\otimes\mathrm{id}_{\mathcal{C}}$ is right exact. Then for any A-left module morphism $f:X\longrightarrow Y$ the cokernel is an A-module morphism because there is a unique morphism μ'_l such that

$\mu_l' \circ (\mathrm{id} \otimes \operatorname{coker} f) = \mu_l' \circ \operatorname{coker}(\mathrm{id} \otimes f) = \operatorname{coker} f \circ \mu_l^Y$. The module properties can be verified without problems. The kernel of an A-module morphism is an A-module morphism even if $A \otimes \mathrm{id}_{\mathcal{C}}$ is not supposed to be right exact. Then one proves easily that the H-left, H-right and H-bimodule categories ${}_H\mathcal{C}$, $\mathcal{C}_H$ and ${}_H\mathcal{C}_H$ respectively are abelian if $\mathcal{C}$ is abelian. If B is a flat bialgebra in $\mathcal{C}$ it follows immediately that ${}^B_B\mathcal{C}$, $\mathcal{C}^B_B$, ${}^B_B\mathcal{C}_B$, ${}_B\mathcal{C}^B_B$,…,${}^B_B\mathcal{C}^B_B$ and $\mathcal{DY}(\mathcal{C})^B_B$ are abelian. Suppose that H is a flat Hopf algebra, then according to Theorem 1.3 the tensor product of the braided abelian category ${}^H_H\mathcal{C}^H_H$ is bi-additive. Similarly it can be proven that $\mathcal{DY}(\mathcal{C})^H_H$ is braided abelian with bi-additive tensor product. Obviously the forgetful functors ${}^H_H\mathcal{C}^H_H \longrightarrow \mathcal{C}$ and $\mathcal{DY}(\mathcal{C})^H_H \longrightarrow \mathcal{C}$ are exact. The flatness of H implies the exactness of the functor $H \ltimes (\_)$. To prove that ${}_H(\_)$ is exact we consider an H-Hopf bimodule morphism g. It holds $g \circ \mathrm{i} \circ \ker(\mathrm{p} \circ g \circ \mathrm{i}) = 0$ and $\mathrm{p} \circ g \circ \mathrm{i} \circ (\mathrm{p} \circ \ker g \circ \mathrm{i}) = 0$ where i and p are the universal morphisms coming from the construction of ${}_H(\_)$. Hence there are unique morphisms $k : \operatorname{Ker}({}_H g) \longrightarrow \operatorname{Ker} g$ and $l : {}_H(\operatorname{Ker} g) \longrightarrow \operatorname{Ker}({}_H g)$ such that $\ker({}_H g) = \mathrm{p} \circ g \circ k$ and ${}_H(\ker g) = \ker({}_H g) \circ l$. Therefore $\mathrm{p} \circ k \circ l = \mathrm{id}$ and $l \circ \mathrm{p} \circ k = \mathrm{id}$ which yields $\ker({}_H g) = {}_H(\ker g)$ up to isomorphism. The dual result for the cokernels will be derived in a similar way. ■

Corollary A.2 *Let $\mathcal{C}$ be a braided abelian category with bi-additive tensor product. Assume that H is a flat Hopf algebra with invertible antipode in $\mathcal{C}$. If X is an H-Hopf bimodule and $X \otimes \mathrm{id}_{\mathcal{C}}$ is left or right exact in $\mathcal{C}$, then $X \otimes_H \mathrm{id}_{\mathcal{C}}$ is left or right exact respectively. Conversely if $X \otimes_H \mathrm{id}_{\mathcal{C}}$ is left/right exact then ${}_H X \otimes \mathrm{id}_{\mathcal{C}}$ is left/right exact in $\mathcal{C}$.*

Proof. Form the proof of Proposition A.1 we know that ${}_H(\ker f) = {}_Y\mathrm{p} \circ \ker f \circ {}_X\mathrm{i}$ for every H-Hopf bimodule morphism $f : X \longrightarrow Y$. It follows that $X \otimes_H \mathrm{id}_{\mathcal{C}}$ is left/right exact when $X \otimes \mathrm{id}_{\mathcal{C}}$ is left/right exact. Suppose that $\mathrm{id}_{\mathcal{C}} \otimes_H X$ is left/right exact. Then Theorem 1.3 implies that $\mathrm{id}_{\mathcal{C}} \otimes_H X$ is left/right exact. ■

Proposition A.3 *If $\mathcal{C}$ is $\otimes$-factor braided abelian and H is a flat Hopf algebra in $\mathcal{C}$ with isomorphic antipode, then ${}^H_H\mathcal{C}^H_H$ and $\mathcal{DY}(\mathcal{C})^H_H$ are $\otimes$-factor braided abelian. The categories ${}^H_H\mathcal{C}^H_H$ and $\mathcal{DY}(\mathcal{C})^H_H$ are $\otimes$-exact braided abelian if $\mathcal{C}$ is $\otimes$-exact braided abelian.*

Proof. For the proof of the $\otimes$-factor abelian property we remark that the projection $\Pi := \mathrm{p} \circ \mathrm{i}$ commutes with Hopf bimodule morphisms. Now suppose that $\mathcal{C}$ is $\otimes$-exact abelian. Let f_1 and f_2 be epimorphisms and g and h be morphisms in ${}^H_H\mathcal{C}^H_H$ such that $g \circ (\mathrm{id} \otimes_H f_2) = h \circ (f_1 \otimes_H \mathrm{id})$. Using Theorem 1.3 we obtain $g \circ (\mathrm{id} \otimes \mathrm{p}) \circ (\mathrm{id} \otimes f_2) = h \circ (\mathrm{id} \otimes \mathrm{p}') \circ (f_1 \otimes \mathrm{id})$ where p and p′ are the corresponding universal morphisms of coinvariants. Since $\mathcal{C}$ is $\otimes$-exact abelian there is a unique morphism t in $\mathcal{C}$ obeying the relation

$$\begin{aligned} g \circ (\mathrm{id} \otimes \mathrm{p}) &= t \circ (f_1 \otimes \mathrm{id}) \,, \\ h \circ (\mathrm{id} \otimes \mathrm{p}') &= t \circ (\mathrm{id} \otimes f_2) \,. \end{aligned} \tag{A.1}$$

f_2 is an H-Hopf bimodule morphism. Thus there is the unique morphism $t \circ (\mathrm{id} \otimes \mathrm{i})$ which fulfills the equations

$$\begin{aligned} g &= t \circ (\mathrm{id} \otimes \mathrm{i}) \circ (f_1 \otimes_H \mathrm{id}) \,, \\ h &= t \circ (\mathrm{id} \otimes \mathrm{i}) \circ (\mathrm{id} \otimes_H f_2) \,. \end{aligned} \tag{A.2}$$

The morphisms $f_1 \otimes_H \mathrm{id}$ and $\mathrm{id} \otimes_H f_2$ are epimorphic in $\mathcal{C}$. Therefore it follows that $t \circ (\mathrm{id} \otimes \mathrm{i})$ is a morphism in ${}^H_H\mathcal{C}^H_H$. In an analogous manner the pull-back condition of Definition 2.5 can be proved for ${}^H_H\mathcal{C}^H_H$. Similarly one verifies that $\mathcal{DY}(\mathcal{C})^H_H$ is $\otimes$-exact abelian. ■

In the next proposition we derive a result which is useful for the study of graded (co-)modules, Hopf bimodules and crossed modules.

Proposition A.4 *Let B be a bialgebra in the $\otimes$-factor braided abelian category $\mathcal{C}$ and suppose that the functor $B \otimes \mathrm{id}_{\mathcal{C}}$ is left exact. Then there are canonical braided monoidal isomorphisms of comodule categories according to*

$$^{B}(\mathcal{C}^{\mathrm{I}}) \cong (^{B}\mathcal{C})^{\mathrm{I}} \quad \textit{and} \quad (^{B}\mathcal{C}^{\mathrm{I}_c}) \cong (^{B}\mathcal{C})^{\mathrm{I}_c} \,. \tag{A.3}$$

Analogous isomorphisms are obtained for the categories of right B-comodules and B-bicomodules. If B is right exact the corresponding results are obtained for the module categories. Suppose that H is a flat Hopf algebra with invertible antipode in $\mathcal{C}$, then there are exact braided monoidal isomorphisms of $\otimes$-factor braided abelian categories

$$\begin{array}{ccc} {}^H_H(\mathcal{C}^{\mathrm{I}})^H_H \cong ({}^H_H\mathcal{C}^H_H)^{\mathrm{I}} \,, & & {}^H_H(\mathcal{C}^{\mathrm{I}_c})^H_H \cong ({}^H_H\mathcal{C}^H_H)^{\mathrm{I}_c} \,, \\ \mathcal{DY}(\mathcal{C}^{\mathrm{I}})^H_H \cong (\mathcal{DY}(\mathcal{C})^H_H)^{I} \,, & \textit{and} & \mathcal{DY}(\mathcal{C}^{\mathrm{I}_c})^H_H \cong (\mathcal{DY}(\mathcal{C})^H_H)^{\mathrm{I}_c} \,. \end{array} \tag{A.4}$$

Proof. We use (3.12) to imbed the catgory $\mathcal{C}$ into $\mathcal{C}^{\mathrm{I}}$ or $\mathcal{C}^{\mathrm{I}_c}$. Then we consider the functor $T : (^{B}\mathcal{C})^{\underline{\mathbb{N}}} \longrightarrow {}^{B}(\mathcal{C}^{\underline{\mathbb{N}}})$, $(X_n, \nu_n)_n \mapsto ((X_n)_n, (\nu_n)_n)$, $\hat{f} \mapsto \hat{f}$. We complete the proof using Propositions A.1, A.3 and Corollary A.2. Similarly all other cases of (co-)modules, Hopf bimodules and crossed modules can be treated. ■

A derivation of [BD1] is the following lemma.

Lemma A.5 *Let H be a Hopf algebra with invertible antipode in the braided category $\mathcal{C}$ with split idempotents, and let H^{ad} be the H-right crossed module defined in Example 1. Then the diagram of H-Hopf bimodule morphisms*

$$\begin{tikzcd} H \otimes H \arrow[r, "(\mathrm{m} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta)"] \arrow[d, "\mathrm{m}"'] & H \ltimes H^{\mathrm{ad}} \arrow[d, "H \ltimes (\varepsilon)"] \\ H \arrow[r] & H \ltimes \mathbf{1} \end{tikzcd} \tag{A.5}$$

is commutative. The horizontal morphisms are isomorphisms.

Proof. Straightforward. We only note that the inverse of $(\mathrm{m} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta)$ is given by $(\mathrm{m} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes S \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta)$. ■

We recall the definition and results on the comma category $(\mathbf{1} \downarrow \mathcal{C})$ [Mac]. It is the category which is formed by objects (X, x) where X is an object in $\mathcal{C}$ and $x : \mathbf{1} \longrightarrow X$ is a morphism in $\mathcal{C}$. The morphisms $f : (X, x) \longrightarrow (Y, y)$ in $(\mathbf{1} \downarrow \mathcal{C})$ are morphisms $f : X \longrightarrow Y$ in $\mathcal{C}$ such that $f \circ x = y$. Without problems the next lemma can be proved.

Lemma A.6 *Suppose that $\mathcal{C}$ is braided with split idempotents. Then the category $(\mathbf{1} \downarrow \mathcal{C})$ is braided with split idempotents. The unit object is given by $(\mathbf{1}, \mathrm{id}_{\mathbf{1}})$, the tensor product of two objects equals $(X, x) \otimes (Y, y) = (X \otimes Y, x \otimes y)$. The tensor product of two morphisms f and g is given by the tensor product $f \otimes g$ in $\mathcal{C}$. The braiding in $(\mathbf{1} \downarrow \mathcal{C})$ is defined through $\Psi^{(\mathbf{1}\downarrow\mathcal{C})}_{(X,x),(Y,y)} = \Psi_{X,Y}$.* ■

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